STIEFEL-WHITNEY NUMBERS AND MAPS
COBORDANT TO EMBEEDINGS

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ABSTRACT. A necessary and sufficient condition is given for a continuous map between compact differentiable manifolds to be cobordant in the sense of Stong to an embedding. For the case of a map \( f: M^n \to S^{n+k} \) the condition reduces to the vanishing of all Stiefel-Whitney numbers of \( M^n \) that involve \( w_i \) for \( i \geq k \).

1. Introduction. A necessary condition for the existence of an embedding of a compact differentiable manifold \( M^n \) in a euclidean space \( R^{n+k} \) (or a sphere \( S^{n+k} \)) is the vanishing of the dual Stiefel-Whitney classes \( w_i(M^n) \in H^i(M^n; \mathbb{Z}/2\mathbb{Z}) \) for \( i \geq k \). This condition is far from sufficient. For example, if \( M^n \) is a real projective \( n \)-space \( P^n \) with \( n = 2^s - 1 \) \((s \geq 4)\), then \( w_i(P^n) = 0 \) for all \( i > 0 \), but \( P^n \) does not embed in \( R^{n+k} \) if \( k < n/4 \). (See \([1, p. 131]\).) However, one can still look for some statement involving embeddings that is implied by the condition \( w_i(M^n) = 0 \) for \( i \geq k \). Because Stiefel-Whitney numbers form a complete system of invariants for certain cobordism theories one can expect a result involving cobordism, and in \([2]\) we have shown that if \( w_i(M^n) = 0 \) for \( i \geq k \) then \( M^n \) is cobordant to a manifold \( M^n_1 \) that embeds in \( S^{n+k} \) provided that \( k \) is not much smaller than \( n \). Equivalently, under the same conditions, any map \( f: M^n \to S^{n+k} \) is bordant to an embedding \( f_1: M^n_1 \to S^{n+k} \).

In this paper we use the notion of cobordism of maps due to Stong and prove that the vanishing of all Stiefel-Whitney numbers of \( M^n \) involving \( w_i(M^n) \) \((i \geq k)\) is necessary and sufficient for a map \( f: M^n \to S^{n+k} \) to be cobordant as a map to an embedding \( f_1: M^n_1 \to S^{n+k}_1 \). In other words if you weaken the original embedding problem using cobordism of maps, then the whole story is told by the usual dual Stiefel-Whitney numbers.

Actually we solve the more general problem of determining when a map \( f: M^n \to S^{n+k} \) is cobordant to an embedding \( f_1: M^n_1 \to S^{n+k}_1 \). In the next
section we develop some necessary conditions involving Stiefel-Whitney numbers of \( f \), and in the third section we state the theorems and prove sufficiency of our conditions. Here the proof is based on a construction suggested to me by Stong. The last section is devoted to examples and remarks. Throughout we use homology and cohomology with \( \mathbb{Z}/2\mathbb{Z} \) coefficients.

2. Stiefel-Whitney numbers of maps. Given a map \( f: M^n \to N^{n+k} \), we have induced maps \( f^* \) and \( f_* \) in cohomology. Recall that if \( x \in H^i(M^n) \) then \( f_*^*(x) = D_N f_*^*(x \cap [M]) \in H^{i+k}(N^{n+k}) \), where \( D_N \) denotes Poincaré duality for \( N^{n+k} \), and where \( f_* \) is also used to denote the induced homology map of \( f \). Then Stong [5] shows that the cobordism class of \( f \) is completely determined by the Stiefel-Whitney numbers of \( f \), namely the numbers

\[
\langle w_{\omega}(N) \cdot f_*^* w_{\omega_1}(M) \cdot \ldots \cdot f_*^* w_{\omega_r}(M), [N]\rangle.
\]

Here \([N]\) denotes the fundamental homology class in \( H^{n+k}(N) \), \( \omega = (i_1, \ldots, i_p) \), \( w_{\omega} = w_{i_1} \cdot \ldots \cdot w_{i_p} \), \( |\omega| = i_1 + \ldots + i_p \), and \( |\omega| + \sum_{j=1}^r (|\omega_j| + k) = n + k \).

We find it convenient to rewrite those numbers with \( r > 0 \) so as to have classes in \( H^*(M) \) evaluated on \([M]\). Observe that

\[
\langle a \cdot f_*^*(b) \cdot f_*^*(c), [N]\rangle = \langle a \cdot f_*^*(b), f_*^*(c \cap [M]) \rangle = \langle f^*(a) \cdot f^*(b), c \cap [M]\rangle = \langle f^*(a) \cdot f^*(b), c, [M]\rangle.
\]

Thus the numbers of \( f \) with \( r > 0 \) take the form

\[
\langle f_*^* w_{\omega}(N) \cdot f_*^* w_{\omega_1}(M) \cdot \ldots \cdot f_*^* w_{\omega_r}(M), [N]\rangle.
\]

Now suppose that \( f \) is an embedding with normal bundle \( \nu \). Then \( f_*^* a = a \cdot w_k(\nu) \). (This is because \( f_*^* \) has another interpretation, namely, \( f_*^* = c^* \Phi \), where \( c: N \to T(\nu) \) is the collapsing map of \( N \) onto the Thom space of \( \nu \), and \( \Phi: H^*(M) \to H^{*+k}(T(\nu)) \) is the Thom isomorphism. If \( i: M \to T(\nu) \) is the inclusion of the zero section then \( f_*^* a = f^* c^* \Phi(a) = i^* \Phi(a) \), and finally \( i^* \Phi(a) = a \cdot w_k(\nu) \) by a basic property of the Thom isomorphism.) To see how \( w_k(\nu) \) is determined by \( f \), let \( N \) be embedded in a euclidean space \( R^i \) with normal bundle \( \eta \). Then \( rM^n \oplus \nu \oplus f^{-1} \eta \) is a trivial bundle, so \( u(M)u(\nu) f^* \overline{w}(N) = 1 \), and hence \( u(\nu) = \overline{w}(M)f^* u(N) \). Note that \( w_i(\nu) = 0 \) if \( i > k \) because \( \nu \) is a \( k \)-dimensional bundle.

Definition. \( \overline{w}(f) \) for any map \( f: M^n \to N^{n+k} \) is defined by \( \overline{w}(f) = \overline{w}(M)f^* \overline{w}(N) \).
We now have a necessary condition for \( f \) to be cobordant to an embedding, namely that \( \tilde{w}_i(f) \) should be zero in numbers if \( i > k \), and that the numbers of the form (1) should be equal to the numbers of the form (2) below:

\[
(2) \quad (f^* w_\omega(N) \cdot w_\omega(M) \cdot \ldots \cdot w_\omega(M) \cdot (\tilde{w}_k(f))^{r-1}, [M]).
\]

3. The main results.

**Theorem.** A map \( f: M^n \rightarrow N^{n+k} \) \((k > 0)\) is cobordant to an embedding \( f_1: M^n_1 \rightarrow N^{n+k}_1 \) if and only if the following conditions hold:

(i) All Stiefel-Whitney numbers of \( f \) involving \( w_\omega(f) \) for \( i > k \) are zero.

(ii) All Stiefel-Whitney numbers of \( f \) as given by (1) are equal to the corresponding Stiefel-Whitney numbers as given by (2).

**Proof.** We have just shown that (i) and (ii) are necessary, so now let \( f \) be a map that satisfies (i) and (ii). We wish to construct a cobordant embedding \( f_1 \). If \( t \) is large, the map \( (f, 0): M^n \rightarrow N^{n+k} \times \mathbb{R}^t \) is homotopic to an embedding with normal bundle \( \eta \) classified by a map \( \eta: M^n \rightarrow BO \). Because \( rM \oplus \eta \cong f^{-1}rN \oplus \tau \), we see that \( w(\eta) = \tilde{w}(M) \cdot f^* w(N) = \tilde{w}(f) \). It follows that the Stiefel-Whitney numbers of the map \( \tilde{\eta} \) which are used to determine the bordism class of this map are a subset of the Stiefel-Whitney numbers of the map \( f \), and the condition (i) of the hypotheses implies that all Stiefel-Whitney numbers of \( \tilde{\eta} \) involving \( w_\omega(\eta) \) for \( i > k \) are zero. Hence \( \tilde{\eta} \) is bordant to a map that factors through \( BO(k) \), say \( \tilde{\eta}_1: M^n \rightarrow BO(k) \subset BO \) with associated bundle \( \eta_1 \) over \( M^n_1 \). (See [3, 17.3, p. 48].) Let \( S(\eta_1 \oplus 1) \) denote the sphere bundle of \( \eta_1 \oplus 1 \) over \( M_1 \), let \( N^{n+k}_1 = S(\eta_1 \oplus 1) \cup N^{n+k} \), and let \( f_1: M^n_1 \rightarrow N^{n+k}_1 \) be the inclusion of the cross-section determined by the trivial line bundle. (I wish to thank R. E. Stong for showing me this construction in the case where \( N^{n+k} = S^{n+k} \).) Then \( f_1 \) is a differentiable embedding, and it remains to show that \( f \) and \( f_1 \) are cobordant. For this purpose we compute the Stiefel-Whitney numbers of \( f_1 \) and compare them with those of \( f \).

Because of the section, \( H^*(M^n_1) \) is a direct summand of \( H^*(S(\eta_1 \oplus 1)) \) so the Serre spectral sequence for \( H^*(S(\eta_1 \oplus 1)) \) collapses and \( H^*(S(\eta_1 \oplus 1)) \) is isomorphic to \( H^*(M^n_1) \oplus H^*(S^k) \). If \( p: S_1 = S(\eta_1 \oplus 1) \rightarrow M_1 \) is the projection, then \( rS_1 = p^{-1}rM_1 \oplus \phi \), where \( \phi \) is the bundle along the fibres. Thus

\[
f_1^*w(N_1) = w(M_1) \cdot w(\eta_1) \quad \text{and} \quad w(\eta_1) = \tilde{w}(M_1)f^*w(N_1) = \tilde{w}(f_1).
\]

Now we are ready to compare numbers. First the numbers of \( f \) and \( f_1 \) with \( r = 0 \) both equal the numbers of \( N \) because \( S(\eta_1 \oplus 1) \) is a boundary.
If \( r > 0 \), the numbers of \( f \) and \( f_1 \) of the form (1) reduce to those of the form (2) by hypothesis (ii) for \( f \) and by construction (i.e., \( f_1 \) is an embedding) for \( f_1 \). Now we use the fact that \( f^* u(N) = u(M) \cdot \overline{w}(f) = u(M)u(\eta) \) and the analogous fact for \( f_1 \) to rewrite the number of the form (2) into the form 
\[
(\omega_\omega(M)w_\omega, (\eta), [M])
\]
with an analogous expression for \( f_1 \). But now we are looking at Stiefel-Whitney numbers of \( \overline{\eta} \) and of \( \overline{\eta}_1 \) and these are equal because \( \overline{\eta} \) and \( \overline{\eta}_1 \) are bordant maps. This completes the proof.

**Corollary.** A map \( M^n \to S^{n+k} \) \((k > 0)\) is cobordant to an embedding \( f_1: M^n \to N^{n+k} \) if and only if all Stiefel-Whitney numbers of \( M^n \) involving \( \overline{w}_i(M^n) \) \((i \geq k)\) are zero. (In interpreting the statement of the Corollary it helps to note that all maps \( f: M^n \to S^{n+k} \) \((k > 0)\) are cobordant.)

**Proof.** Taking \( N^{n+k} = S^{n+k} \) in the Theorem we find that \( u(S^{n+k}) = 1 \) and that \( f^* f_*(x) = 0 \) for all \( x \in H^*(M) \). Thus condition (i) is equivalent to saying that all Stiefel-Whitney numbers involving \( \overline{w}_i(M^n) \) \((i > k)\) should vanish, and condition (ii) is then equivalent to saying that all numbers involving \( \overline{w}_k(M^n) \) should vanish.

4. Applications and examples. If we apply the Corollary to a product we obtain the following result.

**Proposition 1.** If maps \( M^m \to S^{m+p}, N^n \to S^{n+q} \) are both cobordant to embeddings, then any map \( M^m \times N^n \to S^{m+n+p+q-1} \) is cobordant to an embedding. In other words, products always embed better (modulo map cobordism) than the product embedding of the factors. We assume \( p > 0, q > 0 \).

**Proof.** This follows from the Corollary because the top nonzero class (in numbers) of \( M^m \times N^n \) is
\[
\overline{w}_{p+q-2}(M^m \times N^n) = \overline{w}_{p+q-1}(M^m) \cdot \overline{w}_{q-1}(N^n).
\]

**Remark.** One can ask whether the proposition is true without the “modulo map cobordism” clause. In many cases it does hold. For let \( d(X) \) denote the difference between the best euclidean immersion and best euclidean embedding of the manifold \( X \). If \( d(M^m) > 0 \) and \( m \leq n + p + q - 2 \), then we can embed \( M^m \times N^n \) in \( R^{m+n+p+q-1} \) given embeddings of \( M^m \) in \( R^{m+p} \) and \( N^n \) in \( R^{n+q} \). (See [4, p. 319].) But by [4, pp. 320, 321] the product embedding of \((CP^2)^2\) is best possible. However this manifold is cobordant to \((RP^2)^4\) whose product embedding is not best. Hence “modulo map cobordism” cannot be deleted but might possibly be improved to “modulo bordism”.

Now consider the case \( M^n = P^n \), a real projective \( n \)-space. Recall that if \( \alpha \) generates \( H^1(P^n) \) then \( H^*(P^n) = (Z/2Z)[\alpha]/(\alpha^{n+1}) \) and \( u(P^n) = \)
If \( n \) is odd then all Stiefel-Whitney numbers of \( P^n \) are zero and a map \( P^n \to S^{n+k} \) \((k > 0)\) is cobordant to the obvious embedding \( S^n \subset S^{n+k} \). So let \( n \) be even and write \( n = 2^a + b \) with \( 0 \leq b < 2^a \).

**Proposition 2.** A map \( f: P^n \to S^{n+k} \) \((n \text{ even}, k > 0)\) is cobordant to an embedding if and only if \( k \geq n - 2b \).

**Proof.** \( \bar{w}(P^n) = (1 + \alpha)^{-n-1} = (1 + \alpha)^{-2^a+1}(1 + \alpha)^{2^a-b-1} = (1 + \alpha^{2^a+1})^{-1}(1 + \alpha)^{2^a-b-1} = (1 + \alpha)^{2^a-b-1}. \) Let \( p = 2^a - b = n - 2b \). Then \( \bar{w}_i(P^n) = 0 \) if \( i > p \) but the Stiefel-Whitney number \( w^{n-p+1}_i \bar{w}_{p-1} \neq 0 \).

**Remark.** The extreme cases are \( n = 2^a \) and \( n = 2^a+1 - 2 \). In the first case we get \( k \geq n \) and this shows that a high codimension may be needed even for embeddings modulo map cobordism. In the second case we get \( k \geq 2 \). An example of a codimension 2 embedding \( f_1 \) cobordant to \( f \) may be constructed as follows. Let \( M^n_1 = P^n \), let \( N^{n+2}_1 = P^{n+1} \times S^1 \), and let \( f_1 \) be the inclusion of \( P^n \) into \( P^{n+1} \times \{1\} \). Then the normal bundle of \( f_1 \) admits a section, so \( \bar{w}_2(f_1) = 0 \), and hence \( f/f_1(x) = 0 \) for all \( x \). Also \( u(N^{n+2}_1) = 1 \). Thus the Stiefel-Whitney numbers of \( f_1 \) reduce to those of \( P^n \) and the same is true for the Stiefel-Whitney numbers of \( f \).

**Proposition 3.** Let \( f: P^n \to P^{n+k} \) \((k > 0)\) be a map. If \( f^*(\alpha) \neq 0 \) then \( f \) is cobordant to the inclusion \( P^n \subset P^{n+k} \). If \( f^*(\alpha) = 0 \) then \( f \) is cobordant to an embedding if and only if \( n \) is odd or \( k > n - 2b \) where \( n = 2^a + b \) as above.

**Proof.** If \( f^*(\alpha) = \alpha \) (with the obvious abuse of notation) then \( f \) has the same Stiefel-Whitney numbers as the inclusion \( P^n \subset P^{n+k} \). If \( f^*(\alpha) = 0 \) then \( f/f_*(x) = 0 \) for all \( x \), and \( f^*w(P^{n+k}) = 1 \). Thus \( f \) has the same Stiefel-Whitney numbers as a map \( P^n \to S^{n+k} \) and Proposition 2 applies.

Finally we mention the homotopy theoretic interpretation of our results. Stong [5] has shown that cobordism classes of maps from \( n \)-manifolds to \((n + k)\)-manifolds are in 1-1 correspondence with the bordism group \( \Omega^sMO(k + s) \), \( s \) large. On the other hand, \( \Omega^{n+k}_s(MO(k)) \) represents cobordism classes of embeddings of codimension \( k \). The obvious map \( BO(k) \to BO(k + s) \) yields a map \( \Sigma^sMO(k) \to MO(k + s) \) and hence a map \( \Psi: MO(k) \to \Omega^sMO(k + s) \). The induced homomorphism \( \Psi_* \) on bordism is injective.

(For the Stiefel-Whitney numbers of a map \( \phi: N^{n+k} \to MO(k) \) can be written in terms of the associated embedding \( f: M^n \to N^{n+k} \) and take the form (2) of \( \S 2 \). They are included among the Stiefel-Whitney numbers of \( f \) considered now as a map.) We have described the image of \( \Omega_k(*) \) in terms of comput-
able invariants. There is another problem whose solution must also be given by Stiefel-Whitney numbers, namely finding which bordism classes of maps $f: M^n \to N^{n+k}$ can be represented by an embedding $f_1: M^n_1 \to N^{n+k}$. In terms of homotopy theory we are trying to describe the image of the set of homotopy classes $[N^{n+k}, MO(k)]$ in the group $[N^{n+k}, \Omega^s MO(k+s)] = \pi_k(N^{n+k}) = \pi_n(N^{n+k})$. (See [3, p. 37].) The appropriate Stiefel-Whitney numbers of $f$ have the form $\langle w_\alpha(M)f^*(x), [M] \rangle$, where $x \in H^*(N)$ need not be a characteristic class of $N$. In the case where $N^{n+k} = S^{n+k}$, these numbers reduce to the numbers of $M^n$ and it would be interesting to know the answer here and to see whether new nonembedding theorems could be deduced using Stiefel-Whitney classes.

REFERENCES


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