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ON SIMULTANEOUS CHEBYSHEV APPROXIMATION IN THE "SUM" NORM

WILLIAM H. LING

ABSTRACT. Let f_1, f_2 be real valued functions on $[a, b]$ and let S be a nonempty family of real valued functions on $[a, b]$. It is shown that the simultaneous approximation of f_1 and f_2 in the "sum" norm by elements of S is, with one restriction, equivalent to the approximation of the arithmetic mean, $(f_1 + f_2)/2$. A complete characterization of best approximations in the "sum" norm is given including results for varisolvent families.

1. Introduction. Let f_1, f_2 be real valued functions defined on the nondegenerate compact real interval $[a, b]$. Let S be a nonempty family of real valued functions on $[a, b]$. For any real valued function f on $[a, b]$, $\|f\|$ shall mean $\sup_{x\in[a,b]} |f(x)|$. We are concerned with approximating f_1 and f_2 simultaneously in the "sum" norm by elements of S, i.e. we wish to minimize the expression $\| |f_1 - s| + |f_2 - s| \|$. If there exists an $s^* \in S$ such that

$$\| |f_1 - s^*| + |f_2 - s^*| \| = \inf_{s\in S} \| |f_1 - s| + |f_2 - s| \|,$$

we say that s^* is a best simultaneous approximation to f_1 and f_2 in the "sum" norm. Also we say $s^* \in S$ is a best approximation to $(f_1 + f_2)/2$ from S if

$$\|(f_1 + f_2)/2 - s^*\| = \inf_{s\in S} \|(f_1 + f_2)/2 - s\|.$$

There have been some recent papers on simultaneous approximation; see [3], [5] and [6]. In [5] and [6], l^p-simultaneous approximation on the direct sum $X \oplus Y$ is discussed, where X and Y are normed linear spaces. The "sum" norm defined above yields a norm on $C[a, b] \oplus C[a, b]$ but it is not discussed in [5] or [6]. In [3], the classical simultaneous problem is shown to be equivalent to approximating the arithmetic mean, $(f_1 + f_2)/2$, of f_1 and f_2 with an additive weight function. We show here that approximating in the "sum" norm is, with one restriction, equivalent to approximating the arithmetic mean.

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2. **Theorem and discussion.** Let ρ_o, ρ and y_o be defined as

$$\rho_o = \inf_{s\in S}\left\|\frac{f_1+f_2}{2} - s\right\|, \qquad \rho = \inf_{s\in S}\| |f_1 - s| + |f_2 - s| \| \quad \text{and} \quad y_o = \|f_2 - f_1\|.$$

We then have the following.

Theorem. *Let ρ_o, ρ and y_o be defined as above. Then it follows that*
(a) *if $\rho_o \geq y_o/2$, then $\rho = 2\rho_o$, and*
(b) *if $\rho_o < y_o/2$, then $\rho = y_o$.*

Proof. The proof is general, not specifying $\rho_o \geq y_o/2$ or $\rho_o < y_o/2$ until the end.

Claim 1. $\rho \geq 2\rho_o$.

Proof. We have

$$\left\|\frac{f_1+f_2}{2} - s\right\| = \left\|\left(\frac{f_1 - s}{2}\right) + \left(\frac{f_2 - s}{2}\right)\right\| \leq \frac{1}{2}\| |f_1 - s| + |f_2 - s| \|.$$

Taking the infimum over S on both sides of this last inequality yields $\rho_o \leq \frac{1}{2}\rho$.

Claim 2. $\rho \geq y_o$.

Proof. For any $s \in S$, we have

$$|f_1(x) - s(x)| + |f_2(x) - s(x)| \geq |f_1(x) - f_2(x)|$$

for all $x \in [a, b]$ which implies that $\| |f_1 - s| + |f_2 - s| \| \geq y_o$. Thus $\rho \geq y_o$.

Now recall that $|a + b| + |a - b| = 2\max\{|a|, |b|\}$ is an identity for all real numbers a and b. Identifying a with $(f_1(x) + f_2(x))/2 - s(x)$ and b with $(f_1(x) - f_2(x))/2$ we may write

$$|f_1(x) - s(x)| + |f_2(x) - s(x)| = 2\max\left\{\left|\frac{f_1(x)+f_2(x)}{2} - s(x)\right|, \left|\frac{f_1(x) - f_2(x)}{2}\right|\right\}.$$

Taking the supremum over x on both sides of this last equality yields

$$\begin{aligned}\| |f_1 - s| + |f_2 - s| \| &= 2\sup_{x\in[a,b]}\left[\max\left\{\left|\frac{f_1(x)+f_2(x)}{2} - s(x)\right|, \left|\frac{f_1(x) - f_2(x)}{2}\right|\right\}\right]\\ &\leq 2\max\left\{\left\|\frac{f_1+f_2}{2} - s\right\|, \left\|\frac{f_1 - f_2}{2}\right\|\right\}\\ &= 2\max\left\{\left\|\frac{f_1+f_2}{2} - s\right\|, \frac{y_o}{2}\right\};\end{aligned}$$

i.e.

$$\| |f_1 - s| + |f_2 - s| \| \leq 2 \max\{\|(f_1 + f_2)/2 - s\|, y_o/2\}.$$

Now taking the infimum over S on both sides of this last inequality gives

$$\rho \leq 2 \inf_{s \in S} \max \left\{ \left\| \frac{f_1 + f_2}{2} - s \right\|, \frac{y_o}{2} \right\}$$

$$= 2 \max \left\{ \inf_{s \in S} \left\| \frac{f_1 + f_2}{2} - s \right\|, \frac{y_o}{2} \right\} = 2 \max\{\rho_o, y_o/2\}.$$

It follows that

$$\text{(A)} \qquad \rho \leq 2 \max\{\rho_o, y_o/2\}.$$

The proof is now complete. For part (a) of the Theorem, if $\rho_o \geq y_o/2$, inequality (A) combined with Claim 1, ensures $\rho = 2\rho_o$. For part (b), if $\rho_o < y_o/2$, inequality (A) combined with Claim 2, ensures $\rho = y_o$. Q.E.D.

The Theorem plus a short argument by contradiction gives the result that if $\rho_o \geq y_o/2$, then s^* is a best simultaneous approximation to f_1, f_2 in the "sum" norm iff s^* is a best approximation to $(f_1 + f_2)/2$ from S. In a classical setting, one might have f_1, f_2 continuous on $[a, b]$ with S a varisolvent family on $[a, b]$. In such a situation, if the best simultaneous approximation s^* should exist, it would be unique; namely the unique best approximation to $(f_1 + f_2)/2$ from S. (For the definition of varisolvency see [7].)

On the other hand, if $\rho_o < y_o/2$, the Theorem plus a short argument shows that if B_A is the set of best approximations in the "sum" norm, i.e. if $B_A = \{s \in S: \| |f_1 - s| + |f_2 - s| \| = \rho\}$, then $B_A = \{s \in S: \|(f_1 + f_2)/2 - s\| \leq y_o/2\}$. In particular, if S is a varisolvent family and there exists an element $s^* \in B_A$ with $\|(f_1 + f_2)/2 - s^*\| < y_o/2$, varisolvency implies that B_A contains an infinite number of functions.

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DEPARTMENT OF MATHEMATICS, UNION COLLEGE, SCHENECTADY, NEW YORK 12308