ON COMMUTATIVE POWER-ASSOCIATIVE NILALGEBRAS OF LOW DIMENSION

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ABSTRACT. Commutative power-associative nilalgebras of dimension 4 and characteristic ≠ 2 are shown to be nilpotent and all their isomorphism classes are determined.

The long-standing conjecture, originally due to A. A. Albert, that a commutative power-associative nilalgebra of finite dimension over a field is nilpotent, has recently been disproved by Suttles [4], who gave a counterexample of dimension 5. This dimension is generally the best possible, for we show here that every commutative power-associative nilalgebra of dimension 4 over a field of characteristic ≠ 2 is nilpotent, and we determine the isomorphism classes of all such algebras. The proof is elementary and cannot be significantly simplified by the use of the general results of [1] and [2], which require, moreover, additional assumptions about the characteristic.

Throughout, A will denote a commutative power-associative nilalgebra of dimension 4 over a field F of characteristic ≠ 2. The subspace of A generated by elements u, v, w, ⋯ will be denoted by (u, v, w, ⋯). Since every x ∈ A is nilpotent, the powers of x are linearly independent, so we must have x^n = 0 for n ≥ 5; the least n such that x^n = 0 for all x ∈ A is called the nil-index. Now the product of any two elements of A can be written as a linear combination of squares, for xy = (x + y)^2 - x^2 - y^2]. Therefore, if the nilindex is 2, then every product vanishes. If it is 5, then A = (x, x^2, x^3, x^4) for any x with x^4 ≠ 0. These are trivial cases, in each of which there is, up to isomorphism, a unique, associative, algebra. Only the cases of nilindex 3 and 4 are of interest.

1. Nilindex 3. Linearizing the identity (x^2)x = 0 yields
2[(xy)z + (yz)x + (zx)y] = 0 for all x, y, z ∈ A. Therefore, denoting right...
multiplication by \( x \) by \( R_x \) we have, for all \( y, z \in A \),
\[
R_y R_z + R_z R_y = - R_{yz}.
\]
Setting \( y = z = x \) gives
\[
(1) \quad R_x^2 = -2 R_x,
\]
Setting \( y = x, z = x^2 \) in (1), and noting that \( x^3 = 0 \), gives
\[
R_x^3 = 0,
\]
which with (2) implies that \( R_x^3 = 0 \). Now choose any \( x \in A \) with \( x^2 \neq 0 \),
and set \( X = (x, x^2) \). This is carried into itself by \( R_x \), which therefore operates
on the two-dimensional quotient \( A/X \). As \( R_x \) is nilpotent, we have
\[
R_x^2(A/X) = 0,
\]
so \( (xy)x \in X \) for all \( y \in A \), i.e., \( (yx)x = ax + \beta x^2 \) for some \( a, \beta \in F \). Since \( R_x^3 = 0 \), multiplying by \( x \) shows that \( a = 0 \), after which using
(2) and the fact that \( R_y \) is also nilpotent shows \( \beta = 0 \) also. Thus \( yx^2 = 0 \),
and since every product is a linear combination of squares, this shows that
the product of any three elements of \( A \) is zero. In particular, \( A \) is associ-ative, so we have

**Theorem 1.** A commutative power-associative nilalgebra \( A \) of nilindex 3
and of dimension 4 over a field \( F \) of characteristic \( \neq 2 \) is associative, and
\( A^3 = 0 \).

These algebras being associative, their classification is well known;
cf. Kruse and Price [3, Chapter VI]: If \( \dim A^2 = 1 \), then one defines a sym-
metric bilinear form on the 3-dimensional space \( A/A^2 \) by choosing any \( x \)
with \( x^2 \neq 0 \) and defining the product of the cosets of \( u, v \in A \) to be \( \alpha \) when-
ever \( uv = \alpha x^2 \); with respect to this form the length of \( x \) itself is clearly 1.
The problem of classifying these algebras up to isomorphism is identical
with that of classifying such forms with the additional condition that there
be a vector of length 1. Unfortunately, this problem is completely solved
only for certain special fields, e.g., the real and complex numbers. If
\( \dim A^2 = 2 \), the only other possibility, then one chooses \( x, y \in A \) such that
\( x^2 \) and \( xy \) span \( A^2 \). Subtracting, if necessary, a multiple of \( x \) from \( y \) one
can, moreover, so choose \( y \) such that \( y^2 = \alpha x^2 \) for some \( \alpha \in F \). It is easy to
check that if one takes any other \( x \) and \( y \) with these properties, then \( \alpha \) is
replaced by \( \alpha c^2 \) for some \( c \neq 0 \). Therefore, denoting the multiplicative
group of \( F \) by \( F^* \), these algebras are parameterized by the elements of
\( F^*/(F^*)^2 \) and 0.

**2. Nilindex 4.** Choose any \( x \) with \( x^3 \neq 0 \) and set \( X = (x, x^2, x^3) \). We
claim \( (x^2, x^3) = A^2 \). It is sufficient to show that \( y^2 \in (x^2, x^3) \) for all \( y \), and
we may further suppose $y \notin X$ and $y^2 \neq 0$, else the matter is trivial. Set $Y = (y, y^2, y^3)$. Then $X \cap Y$ is a proper subalgebra of $X$, hence must be contained in $(x^2, x^3)$, and is a subalgebra of $Y$ of dimension equal to $\dim Y - 1$ (since $\dim X = 3$), and therefore must contain $y^2$; thus $y^2 \in (x^2, x^3)$ as asserted. It follows that $A^2A^2 = 0$. Now $y$ being arbitrary, we have $yx^2 \in A^2$, hence $yx^2 = ax^2 + bx^3$ for some $a, b \in F$. We claim $a = 0$. Otherwise, setting $z = (1/a)(y - bx)$, we have $zx^2 = x^2$; computing $[(z + x^2)^2(z + x^2)] \cdot (z + x^2)$, which must vanish, we find, using the fact that $A^2A^2 = 0$, that it is $2x^2$, a contradiction. If now we replace $x$ by $x + x^2$, thereby replacing $x^2$ by $x^2 + 2x^3$ but leaving $x^3$ unchanged, it follows that $yx^3$ is also a multiple of $x^3$. In fact, $yx^3 = 0$, for if $yx^3 = dx^3$, then computing $[(y + x^3)^2(y + x^3)] \cdot (y + x^3)$, which must vanish, one gets $2d^3x^3$, so $d = 0$. We see now that replacing the original $y$ by $y - bx$, for which we have $(y - bx)x^2 = 0$, one can so choose $y$ such that $y \notin X$ and $yx^2 = yx^3 = 0$, so $yA^2 = 0$. The product of any four of the elements $x, x^2, x^3, y$ vanishes, and as these span $A$, it follows that the product of any 4 elements of $A$ vanishes, so $A^4 = 0$. Therefore, we have

**Theorem 2.** If $A$ is a commutative power-associative nilalgebra of nil-index 4 and of dimension 4 over a field $F$ of characteristic $\neq 2$, then $A^4 = 0$ and there is $y \notin A^2$ such that $yA^2 = 0$.

Theorem 2. If $A$ is a commutative power-associative nilalgebra of nil-index 4 and of dimension 4 over a field $F$ of characteristic $\neq 2$, then $A^4 = 0$ and there is $y \notin A^2$ such that $yA^2 = 0$.

The $y$ of the theorem is not unique, but as $\dim A/A^2 = 2$, there cannot be, modulo $A^2$, two independent elements both annihilating $A^2$, so $y^2$ is determined up to multiplication by an element of $(F^*)^2$. As before, $x$ will denote an element of $A$ such that $x^3 \neq 0$; then clearly $y \notin (x, x^2, x^3)$ so $A = (x, x^2, x^3, y)$. We have the following possibilities:

1. We can so choose $x$ and $y$ such that $yx = 0$. If $y^2 = 0$ then $A$ is unique, the direct sum of $(y)$ and $(x, x^2, x^3)$, and is associative. If $y^2 = \beta x^3$ with $\beta \neq 0$, setting $y' = y/\beta$ and $x' = x/\beta$ gives $y'^2 = x'^3$. This unique algebra is also associative. If $y^2 = \alpha x^2 + \beta x^3$ with $\alpha \neq 0$, replacing $x$ by $x + (\beta/2\alpha)x^2$ shows we may assume $\beta = 0$. Clearly $\alpha$ is determined at most up to multiplication by an element of $(F^*)^2$, and it is easy to check that another choice of $x$ and $y$ replaces $\alpha$ by $\alpha c^2$ for some $c \neq 0$, so we have a family of algebras, all nonassociative, parameterized by the elements of $F^*/(F^*)^2$.

2. We cannot so choose $x$ and $y$ such that $yx = 0$. Therefore we cannot have $yx = \beta x^3$ since $x(y - \beta x^2) = 0$. Choosing any $x$ with $x^3 \neq 0$, we may assume that $yx = yx^2 + \delta x^3$ with $y \neq 0$. Replacing $x$ by $yx + (\delta/2)x^2$ shows we
may so choose \( x \) such that \( yx = x^2 \). If \( y^2 = 0 \) we have a unique nonassociative algebra. If \( y^2 = \beta x^3 \) with \( \beta \neq 0 \), replacing \( x \) by \( x/\beta \) and \( y \) by \( y/\beta \) shows we may assume \( \beta = 1 \) and we have a unique algebra, which is not associative. Finally, if \( y^2 = ax^2 + \beta x^3 \) with \( a \neq 0 \), then as \( y(y - ax) = \beta x^3 \), we must have \( (y - ax)^3 = 0 \) (else we would replace \( x \) by \( y - ax \)); the left side is \( a^2(1 - a)x^3 \), so \( a = 1 \). Replacing \( x \) by \( x + (\beta/2)x^2 \) and \( y \) by \( y + \beta x^2 \), shows we may assume \( \beta = 0 \), so we have a unique algebra given by \( y^2 = yx = x^2 \) and, as always, \( yx^2 = yx^3 = 0 \). It is not associative. This ends the classification. We have found one family of algebras parameterized by \( F*/(F^*)^2 \), and 5 individual algebras of which precisely 2 are associative.

REFERENCES


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