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A SIMPLE PROOF OF WIENER'S $1/f$ THEOREM

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ABSTRACT. We give a simple proof of Wiener's theorem on the reciprocals of absolutely convergent Fourier series.

One of Wiener's famous theorems states that if $f(x)$ has an absolutely convergent Fourier series and never vanishes, then $1/f(x)$ has an absolutely convergent Fourier series. The original proof of this utilized the so-called localization principle which depended in turn on the special nature of the "triangle" and "trapezoid" functions. A beautiful modern proof emerges from the work of Gelfond which involves function algebras and his fundamental notion of maximal ideal spaces. Thus neither of these proofs is particularly simple.

We propose, here, to give a rather simple elementary proof. Though many readers will have no difficulty in recognizing such items as the "spectral norm", the "openness of invertible elements", and other overlaps with older proofs, we do point out that this proof is self-contained and quite direct.

We write, as usual,

$$\left\| \sum_{-\infty}^{\infty} a_n e^{inx} \right\| = \sum_{-\infty}^{\infty} |a_n|,$$

and we recall the triangle inequalities

$$\|f + g\| \leq \|f\| + \|g\|, \qquad \|f \cdot g\| \leq \|f\| \cdot \|g\|.$$

We also will need the inequalities

$$\operatorname{Max}|f| \leq \|f\| \leq \operatorname{Max}|f| + 2 \operatorname{Max}|f'|,$$

the first of which is trivial, while the second follows at once from Schwarz' inequality and Parseval's theorem. We have, namely, writing $f(x) = \Sigma\, a_n e^{inx}$, that $|a_0| \leq \operatorname{Max}|f(x)|$ while

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$$\left(\sum_{n\neq 0} |a_n|\right)^2 \leq \sum_{n\neq 0} \frac{1}{n^2} \cdot \sum_{n\neq 0} n^2|a_n|^2 = \frac{\pi^2}{3} \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(x)|^2\,dx$$

$$\leq (\pi^2/3)\operatorname{Max}|f'(x)|^2 \leq 4\operatorname{Max}|f'(x)|^2$$

so that

$$\sum_{n\neq 0} |a_n| \leq 2\operatorname{Max}|f'(x)|.$$

Now suppose $\|f\| < \infty$ and $f(x)$ has no zeros. We may assume, then, w.l.o.g., that $|f(x)| \geq 1$ for all x. Determine P, a high enough partial sum of f, so that $\|P - f\| \leq 1/3$, and consider the sum

$$S = \sum_{n=1}^{\infty} \frac{(P-f)^{n-1}}{P^n}.$$

We will show that S converges in norm.

We have, namely, $|P(x) - f(x)| \leq 1/3$, so that $|P(x)| \geq |f(x)| - |P(x) - f(x)| \geq 2/3$. Hence $\operatorname{Max}|1/P^n| \leq (3/2)^n$; also $(1/P^n)' = -nP'/P^{n+1}$ so that $\operatorname{Max}|(1/P^n)'| \leq nA(3/2)^{n+1}$, where $A = \operatorname{Max}|P'|$. Thus we have the estimate $\|1/P^n\| \leq (3An+1)(3/2)^n$. Since we also have $\|(P-f)^{n-1}\| \leq \|P-f\|^{n-1} \leq 1/3^{n-1}$, we obtain the bound

$$\|(P-f)^{n-1}/P^n\| \leq (9An+3)/2^n.$$

Thus S does indeed converge in norm. In particular, then, S converges uniformly, and the sum of this geometric series is clearly $1/f(x)$. The Fourier coefficients of $1/f(x)$ can therefore be obtained by term by term integration of the series S, and we conclude that

$$\|1/f(x)\| \leq \sum \|(P-f)^{n-1}/P^n\| < \infty,$$

as required.

In the very same manner, one can prove Lévy's extension of Wiener's theorem, that if F is analytic on the range of $f(x)$, then $F(f(x))$ has an a.c. Fourier series. Namely, one simply considers

$$\sum_{n=0}^{\infty} \frac{F^{(n)}(P(x))}{n!}(f(x) - P(x))^n,$$

and just repeats the very same steps as before.

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