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THE VARIATION OF BROWDER'S ESSENTIAL SPECTRUM

JOHN J. BUONI

ABSTRACT. This paper is devoted to giving various conditions concerning small perturbation of linear operators which imply the continuous variation of the Browder essential spectrum in the Hausdorff topology on the complex plane.

1. **Introduction.** Throughout this paper $\mathcal{H}$ will denote an infinite dimensional complex Hilbert space, $\mathcal{L}(\mathcal{H})$ will represent the algebra of all bounded linear operators on $\mathcal{H}$, $\mathcal{C}(\mathcal{H})$ will be the collection of closed densely defined linear operators on $\mathcal{H}$, and by $\mathcal{K}$ we shall mean the ideal of all compact operators on $\mathcal{H}$. For $T \in \mathcal{C}(\mathcal{H})$, denote the spectrum of T by $\Sigma(T)$. Let π be the canonical projection from $\mathcal{L}(\mathcal{H})$ onto the (Calkin) quotient algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}$. For every $T \in \mathcal{L}(\mathcal{H})$ the spectrum $E(T)$ of $\pi(T)$ in $\mathcal{L}(\mathcal{H})/\mathcal{K}$ will be called the Calkin essential spectrum of T. For $T \in \mathcal{C}(\mathcal{H})$, the Browder essential spectrum, $B(T)$, is defined to be the set of $\lambda \in \Sigma(T)$ such that at least one of the following conditions holds: (1) $R(\lambda - T)$, the range of $\lambda - T$, is not closed; (2) λ is a limit point of $\Sigma(T)$; (3) $\bigcup_{n>0} N[(\lambda - T)^n]$ is infinite dimensional, where $N(A)$ denotes the null space of a linear operator A.

Recall that a closed, densely defined operator T is called a Fredholm operator if and only if $\dim N(T) < \infty$, $R(T)$ is closed, and $\operatorname{codim} R(T) < \infty$. The index of a Fredholm operator T is defined by $j(T) = \dim N(T) - \operatorname{codim} R(T)$. Denote by Φ the collection of all Fredholm operators, and by Φ_0, those Fredholm operators with zero index.

In view of the above notation, we have by Atkinson's theorem for $T \in \mathcal{L}(\mathcal{H})$ that $E(T) = \{\lambda \in \Sigma(T): \lambda - T \notin \Phi\}$. Also, $B(T) = \{\lambda \in \Sigma(T): \lambda - T \notin \Phi_0$ or λ is not an isolated point of $\Sigma(T)\}$. For $T \in \mathcal{L}(\mathcal{H})$, the spectral radius of T shall be denoted by $r_\sigma(T)$, and of $\pi(T)$ by $r_e(T)$. Nussbaum [8] has shown that $r_e(T) = \sup\{|\lambda| \mid \lambda \in B(T)\}$. This latter quantity shall be denoted by $r_b(T)$.

The purpose of this note is to describe various conditions on the convergence of linear operators which imply the continuous variation of the

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2. **Sets with property Γ.** Let $\{T_n\}$ be a sequence in $\mathfrak{L}(\mathfrak{H})$ converging to T in norm. In [6] conditions are given which guarantee the convergence of $\Sigma(T_n)$ to $\Sigma(T)$ in the Hausdorff metric. See also [9, p. 36]. In this section we generalize some of those results to $B(\cdot)$.

The ascent of T, $\alpha(T)$, is the smallest integer p such that $N(T^p) = N(T^{p+1})$, and the descent of S, $\gamma(S)$, is the smallest integer q such that $R(S^q) = R(S^{q+1})$.

D. C. Lay [4] has shown that for $\lambda_0 \in \Sigma(T)$ the following are equivalent to $\lambda_0 \notin B(T)$:

(i) λ_0 is a pole of the resolvent $(\lambda - T)^{-1}$ of finite rank,

(ii) λ_0 has finite ascent, descent and defect,

(iii) $n(\lambda_0 - T) = d(\lambda_0 - T) < \infty$ and $\alpha(\lambda_0 - T) < \infty$.

The upper semicontinuity of $B(\cdot)$ is now formulated. See also [10].

Lemma 2.1. *Let $\{T_n\}$ be a sequence of $\mathfrak{L}(\mathfrak{H})$ converging to T in norm. Then for any open set V containing the origin, there exists N such that for all $n > N$, $B(T_n) \subset B(T) + V$.*

Proof. Assume not. Then by passing to subsequences if necessary, it may be assumed that for each n, there exists an α_n such that $\alpha_n \in B(T_n)$ but $\alpha_n \notin B(T) + V$. Since the α_n's are bounded, and if necessary pass to subsequences, it may be assumed that $\lim_{n\to\infty}\alpha_n = \alpha$. Therefore, $\alpha \notin B(T) + V$, and $\alpha \notin B(T)$. Now by the upper semicontinuity of $\Sigma(\cdot)$ and $E(\cdot)$, it is seen that $\alpha \in \Sigma(T)$ and $\alpha_n \notin E(T_n)$ for all n greater than some N. This yields that α is an isolated point of $\Sigma(T)$ with finite nullity and zero index and, also, that $(\alpha_n - T_n)$ is a Fredholm operator for all $n > N$. Now by upper semicontinuity of the isolated parts of the spectrum [2, IV-3.1] and by the stability of index and nullity for Fredholm operators $((\alpha_n - T_n) \to (\alpha - T))$ [2, IV-5.17], it follows that the α_n's are isolated points of $\Sigma(T_n)$ with finite nullity and zero index. This contradicts the fact that the $\alpha_n \in B(T_n)$ and this yields the Lemma.

Rickart [9, p. 37] has pointed out that if for all $\lambda \notin \Sigma(T_n)$ there exists a real number $k > 0$ such that $kr_\sigma((\lambda - T_n)^{-1}) \geq \|(\lambda - T_n)^{-1}\|$ for all n, then the continuity of the spectrum follows without the need of commutativity, where $r_\sigma(A)$ is the spectral radius of A. With this remark in mind and in view of [6], the following definition is made.

Definition. A subset τ of $\mathfrak{L}(\mathfrak{H})$ has property Γ iff the following hold:

(1) if $T \in \tau$ then $\lambda + T \in \tau$ for all λ;

(2) there exists a number $k > 0$ such that for each $T \in \tau$, if T^{-1} exists then $r_\sigma(T^{-1}) \geq k \, \|T^{-1}\|$;

(3) there exists a number $\hat{k} > 0$ such that for each $T \in \tau$, if $(\pi(T))^{-1}$ exists, then $r_b(T^{-1}) \geq \hat{k}\|(\pi(T))^{-1}\|$.

Theorem 2.2. *Let τ in $\mathfrak{L}(\mathcal{H})$ have property Γ and let $T_n \to T$ in norm, where $T_i \in \tau$. If the isolated points of $\Sigma(T_n)$ converge to the boundary of $\Sigma(T)$, then for any open set V containing the origin there exists N such that for all $n > N$, $B(T_n) \subset B(T) + V$ and $B(T) \subset B(T_n) + V$.*

Proof. Since the first inclusion is given by the preceding Lemma, it remains to prove the second. It may be assumed that V is a circle of radius $2\epsilon > 0$. Now suppose that the inclusion is false. By using the boundedness of $B(T)$ and passing to subsequences, if necessary, we can obtain $\alpha_0 \in B(T)$ such that $|\alpha_0 - \alpha| > \epsilon$ for every $\alpha \in B(T_n)$ and for infinitely many values of n. On the other hand, by the continuity of $\Sigma(\cdot)$ and $E(\cdot)$ on the set τ [6], there exists $\alpha_n \in \Sigma(T_n) - E(T_n)$ such that $\alpha_n \to \alpha_0$ and $\alpha_0 \in B(T) - E(T)$. By our assumption, $\alpha_n \notin B(T_n)$ and is thus an isolated point of $\Sigma(T_n)$. Therefore, by our hypothesis, α_0 lies on the boundary of $\Sigma(T)$. In view of [5, 2.9] and Lay's characterization of the Browder essential spectrum, $\alpha_0 \notin B(T)$. This yields a contradiction.

The following topological example may possibly indicate the futility in attempting to obtain more general results via the same argument.

Example. Set $K_n = \{(x, y) | x^2 + y^2 \leq 1, |y| \geq |x/n|\} \cup \{1/n\}$, where $n \geq 2$. It is clear that K_n converges to the unit disc in the Hausdorff metric. However, the isolated points of K_n converge to an interior point of the disc.

More positive results can be obtained when considering the Weyl essential spectrum, $W(T)$. Recall $W(T) = \{\lambda | \lambda - T \notin \Phi_0\}$ and that $E(T) \subseteq W(T) \subseteq B(T) \subseteq \Sigma(T)$.

Theorem 2.3. *Let $\{T_n\}$ be a sequence of $\mathfrak{L}(\mathcal{H})$ converging to T in norm. Then for any open set V containing the origin, there exists N such that for all $n > N$, $W(T_n) \subset W(T) + V$. Furthermore, if $T_i \in \tau$, where τ has property Γ, then $W(T) \subset W(T_n) + V$.*

Proof. As in the proof of Lemma 2.1, it may be assumed that there exists $\alpha_n \in W(T_n) - E(T_n)$ for each n, $\alpha_0 \notin W(T)$ and $\lim_{n\to\infty} \alpha_n = \alpha_0$. By the stability of the index for Fredholm operators, it follows that the index of $(\alpha_n - T_n)$ is zero for all n greater than some N. This contradicts the fact

that the $\alpha_n \in W(T_n)$ and this yields the first part of the theorem.

Now assume that the second conclusion of the theorem is false. Then we may assume, as in the proof of Theorem 2.2, that there exists $\alpha_n \in \Sigma(T_n) - W(T_n)$ for sufficiently large n, $\alpha_0 \in W(T) - E(T)$ and $\lim_{n\to\infty}\alpha_n = \alpha_0$. Again by the stability of the index for Fredholm operators, it follows that the index of $(\alpha - T)$ is zero. This yields a contradiction and completes the proof of the theorem.

The following is an immediate consequence of above results.

Corollary 2.4. *Let $\{T_n\}$ be a sequence of $\mathfrak{L}(\mathcal{H})$ converging to T in norm, where $T_i \in \tau$ and τ has property Γ. If $W(T) = B(T)$, then for any open set V containing the origin, there exists N such that for all $n > N$, $B(T_n) \subset B(T) + V$ and $B(T) \subset B(T_n) + V$.*

3. **Consequences of generalized convergence.** Let S_M be the unit sphere of M, a closed subspace of the Hilbert space $\mathcal{H}$. For any two closed subspaces M, N of $\mathcal{H}$, we set $\sigma(M, N) = \sup_{u\in S_M} \operatorname{dist}(u, N)$, and $\hat{\sigma}(M, N) = \max[\sigma(M, N), \sigma(N, M)]$, $\sigma(0, N) = 0$ and $\sigma(M, 0) = 1$ where $\operatorname{dist}(u, N)$ is the distance from u to N. $\hat{\sigma}(M, N)$ is called the gap between M and N. The set of all closed subspaces of $\mathcal{H}$ becomes a metric space under $\hat{\sigma}$. For $T, S \in C(\mathcal{H})$, the graphs $G(T)$ and $G(S)$ are closed subspaces of the product space $X \times X$. The distance between T and S can then be measured by the "gap" between the closed subspaces $G(T)$ and $G(S)$. Set $\hat{\sigma}(T, S) = \hat{\sigma}(G(T), G(S))$. Then a sequence of operators T_n converges to T in the generalized sense iff $\hat{\sigma}(T_n, T) \to 0$.

The main result of this section is

Theorem 3.1. *Let Γ be a compact subset of the plane and let $T \in \mathcal{C}(\mathcal{H})$ such that $\Gamma \cap B(T) = \emptyset$. If $\{T_n\}$ is a sequence in $\mathcal{C}(\mathcal{H})$ that converges to T in the generalized sense, then there exists a positive integer k such that $\Gamma \cap B(T_n) = \emptyset$, for every $n \geq k$.*

Proof. Let V be an open neighborhood containing Γ such that its boundary, ∂V, consists of finitely many disjoint closed Jordan curves, $\partial V \cap \Sigma(T) = \emptyset$ and $\overline{V} \cap B(T) = \emptyset$.

Since $\Lambda = V \cap \Sigma(T)$ is a bounded subset whose intersection with $B(T)$ is empty, it follows that the spectral projection P_Λ associated with Λ has finite rank. Therefore there exists a positive integer k such that for every $n \geq k$ we have $\partial V \cap \Sigma(T_n) = \emptyset$ and P_{Λ_n} has finite rank where $\Lambda_n = V \cap \Sigma(T_n)$ [2, IV-3.16]. Therefore $\Lambda_n \cap B(T_n) = \emptyset$ and hence, $\Gamma \cap B(T_n) \leq V \cap B(T_n) = \emptyset$, for every $n \geq k$.

Corollary 3.2. *Let r be any positive number such that the boundary of $V_r = \{\alpha: |\alpha| \leq r\}$ has empty intersection with $B(T)$. Then for any open set V such that $B(T) \subset V \cup V_r'$ there exists k such that $B(T_n) \subset V \cup V_r'$ for all $n > k$.*

Proof. Since the complement of $V \cup V_r'$ is a compact set whose intersection with $B(T)$ is empty, the result follows from Theorem 3.1.

Corollary 3.2 is the biproduct of an attempt to generalize Lemma 2.1 to unbounded operators. An example of Kato [2, III-3.4] indicates that $\Sigma(T)$ is not upper semicontinuous in this sense. This should be compared with [6, p. 173].

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DEPARTMENT OF MATHEMATICS, YOUNGSTOWN STATE UNIVERSITY, YOUNGSTOWN, OHIO 44503