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SPHERICAL DISTRIBUTIONS OF N POINTS WITH MAXIMAL DISTANCE SUMS ARE WELL SPACED

KENNETH B. STOLARSKY

ABSTRACT. It is shown that if N points are placed on the unit sphere in Euclidean 3-space so that the sum of the distances which they determine is maximal, then the distance between any two points is at least $2/3N$. Results for sums of λth powers of distances are also given.

The following problem is open: How should one place N points $p_1, \cdots, p_N$ on the surface of the unit sphere U in E^3 (i.e. on $x^2 + y^2 + z^2 = 1$) so that the sum of the distances which they determine is maximal? Many closely related problems have been studied; see for example [1]–[10]. Here we show that the points must be "well spaced" in the sense that no two can be closer than a distance of $2/3N$. In fact, we will show a bit more. Let $|p_i - p_j|$ denote the *Euclidean* distance between p_i and p_j.

Theorem. *Let* $0 < \lambda < 2$. *If* $p_1, \cdots, p_N$ *are placed on* $x^2 + y^2 + z^2 = 1$ *so that*

$$\sum_{i<j} |p_i - p_j|^\lambda \tag{1}$$

is maximal, then for $i \neq j$ *we have*

$$|p_i - p_j| \geq [4\lambda/2^\lambda(2+\lambda)]^{1/(2-\lambda)} N^{-1/(2-\lambda)}. \tag{2}$$

We can improve the constant here, but it is more important to improve the exponent of N. Since $\min |p_i - p_j| \ll N^{-1/2}$ can be shown easily by the pigeonhole principle for any distribution of N points on this sphere, the exponent is almost best possible for λ small. However, it is possible (and suggested by a result of Björck; see [3, p. 261, Remark 1]) that $|p_i - p_j| \gg N^{-1/2}$ for $0 < \lambda < 2$.

It is interesting to observe that spherical distributions of points with maximal *spherical* distance sums are not necessarily well spaced [6], [8].

For the proof of the Theorem let $p_1, \cdots, p_N$ maximize the sum

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$$\sum_{i<j} G(|p_i - p_j|), \tag{3}$$

where G is a continuous function defined on $[0, 2]$ which has a fourth derivative everywhere in $(0,2)$. We will choose G later. Rotate U so that p_1 coincides with the north pole $(0, 0, 1)$. Let

$$f(p) = \sum_{i=2}^{N} G(|p - p_i|), \tag{4}$$

and let C be a circle on U centered at p_1. Clearly $f(p_1)$ must be at least as large as the average value of $f(p)$ on C. Choose spherical coordinates (θ, ϕ) so that any point $p = (x, y, z) \in U$ is given by $(\sin\phi \sin\theta, \sin\phi\cos\theta, \cos\phi)$. Thus

$$\frac{1}{2\pi}\int_0^{2\pi} f(p)\, d\theta \leq f(0, 0, 1). \tag{5}$$

Next, let $g = g(x, y, z)$ be any function with a convergent power series expansion

$$g(x, y, z) = \sum_{i,j,k} c_{ijk} x^i y^j (z-1)^k \tag{6}$$

about $(0, 0, 1)$. For $(x, y, z) \in C \subseteq U$, one has for ϕ small,

$$\begin{aligned} \Delta &\equiv \frac{1}{2\pi}\int_0^{2\pi} g(x, y, z)\, d\theta - g(0, 0, 1) \\ &= \sum_{i,j,k} c_{ijk} (\sin\phi)^{i+j} (\cos\phi - 1)^k \frac{1}{2\pi}\int_0^{2\pi} \sin^i\theta \cos^j\theta\, d\theta. \end{aligned} \tag{7}$$

Now the integral on the right of (7) is zero unless i and j are both even. Hence when ϕ is small,

$$\Delta = \tfrac{1}{4}\phi^2 \left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} - 2\frac{\partial g}{\partial z}\right)\Bigg|_{p_1} + O(\phi^4). \tag{8}$$

In fact (8) is clearly valid under the weaker assumption that all partial derivatives of g of order at most four exist.

Let $q = (x_0, y_0, z_0) \in U$ be fixed. We now set $g(p) = G(u)$, where $u = |p - q|$. Then

$$\frac{\partial g}{\partial x} = \frac{dG}{du}\frac{(x - x_0)}{|p - q|}$$

and

$$\frac{\partial^2 g}{\partial x^2} = \frac{d^2 G}{du^2}\frac{(x - x_0)^2}{|p - q|^2} + \frac{dG}{du}\frac{(y - y_0)^2 + (z - z_0)^2}{|p - q|^3}.$$

Upon evaluating partials at $p = p_1$, we find that the coefficient of $\tfrac{1}{4}\phi^2$ in (8) is

$$\left(1 - \frac{u^2}{4}\right) \frac{d^2G}{du^2} + \left(\frac{1}{u} - \frac{3}{4}u\right) \frac{dG}{du},$$

where $u = |p_1 - q|$. Thus

$$\Delta = \tfrac{1}{4}\phi^2 \left\{\frac{1}{u} \frac{d}{du} \left[u \left(1 - \frac{u^2}{4}\right) \frac{dG}{du}\right]\right\} + O(\phi^4) = a(u)\phi^2 + O(\phi^4). \tag{9}$$

We now replace q by p_i, where $2 \leq i \leq N$, and set $u_i = |p_1 - p_i|$ so (9) becomes

$$\Delta_i = \tfrac{1}{4} a(u_i)\phi^2 + O(\phi^4). \tag{10}$$

Assume now that the sum (4) is maximal. Let s be the number of points p_i which coincide with p_1, and let every point on C make the small angle ϕ with (0, 0, 1). We apply (5), (7), (9), and (10) with g replaced by f (recall (4)), and obtain

$$\Delta = s \cdot 2\pi \sin \phi + \phi^2 \sum_{i=2}^{N}{}' a(u_i) + O(\phi^4) \leq 0. \tag{11}$$

Here the dashed sum is taken over those i for which $p_i \neq p_1$. By letting $\phi \to 0$ we see first that $s = 0$, so the sum is over all $i \neq 1$. We then see that the sum is at most zero, i.e.

$$\sum_{i=2}^{N} a(|p_1 - p_i|) \leq 0. \tag{12}$$

If $G(u) = u^\lambda$, where $0 < \lambda < 2$, then

$$a(u) = \lambda^2/4u^{2-\lambda} - \lambda(\lambda + 2)u^\lambda/16.$$

Thus for any $j \neq 1$ we have

$$\begin{aligned} \lambda^2|p_1 - p_j|^{\lambda - 2} &\leq \sum_{i=2}^{N} \lambda^2|p_1 - p_i|^{\lambda - 2} \\ &\leq \tfrac{1}{4}\lambda(2 + \lambda) \sum_{i=2}^{N} |p_1 - p_i|^\lambda \leq \tfrac{1}{4}\lambda(2 + \lambda)2^\lambda \cdot N. \end{aligned}$$

Since p_1 was chosen arbitrarily, this completes the proof of the Theorem.

One can now prove many theorems of this type by varying the choice of G. If we take

$$G(u) = \int_0^u th(t) \ln \left\{\frac{u}{t} \frac{(4 - t^2)^{1/2}}{(4 - u^2)^{1/2}}\right\} dt, \tag{13}$$

then a purely formal calculation gives $a(u) = \tfrac{1}{4} h(u)$. Thus if the integral

(13) converges for $h(t)$, we have

$$\sum_{i=2}^{N} h(|p_1 - p_i|) \leq 0 \tag{14}$$

whenever the sum (3) is maximal. In the case $G(u) = u$, we have $h(t) = t^{-1} - \frac{3}{4}t$. We shall now choose $h(t)$ so that (14) is as strong as possible.

It is not hard to show there is a function $w(t)$ continuous on the closed interval $[0, 2]$ such that (13) converges when $\epsilon > 0$ and

$$h(t) = t^{-2}(\ln(1/t))^{-2}(\ln\ln(1/t))^{-1-\epsilon} - w(t). \tag{15}$$

(In the case $G(u) = u$, the role of $w(t)$ was played by $\frac{3}{4}t$.) It follows (from (14); we suppress the details) that if (3) is maximal, then for $i \neq j$ we have

$$|p_i - p_j| \gg N^{-1/2}(\ln N)^{-1}(\ln\ln N)^{-1/2 - 10\epsilon}. \tag{16}$$

Here $G(u)$ behaves like $\epsilon^{-1}(\ln\ln(1/u))^{\epsilon}$ as u approaches 0 from the right. However, we know of no continuous G for which (3) implies $|p_i - p_j| \gg N^{-1/2}$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801

Current address: Department of Mathematics, University of Colorado, Boulder, Colorado 80302