SPHERICAL DISTRIBUTIONS OF N POINTS
WITH MAXIMAL DISTANCE SUMS ARE WELL SPACED
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ABSTRACT. It is shown that if N points are placed on the unit sphere in Euclidean 3-space so that the sum of the distances which they determine is maximal, then the distance between any two points is at least 2/3N. Results for sums of Xth powers of distances are also given.

The following problem is open: How should one place N points p1, • • • , pN on the surface of the unit sphere U in E3 (i.e. on x2 + y2 + z2 = 1) so that the sum of the distances which they determine is maximal? Many closely related problems have been studied; see for example [1]–[10]. Here we show that the points must be “well spaced” in the sense that no two can be closer than a distance of 2/3N. In fact, we will show a bit more. Let |pi - pj| denote the Euclidean distance between pi and pj.

Theorem. Let 0 < λ < 2. If p1, • • • , pN are placed on x2 + y2 + z2 = 1 so that

\[ \sum_{i<j} |p_i - p_j|^\lambda \]

is maximal, then for i ≠ j we have

\[ |p_i - p_j| \geq [4\lambda/2^\lambda(2 + \lambda)]^{1/(2-\lambda)}N^{-1/(2-\lambda)}. \]

We can improve the constant here, but it is more important to improve the exponent of N. Since \( \min |p_i - p_j| \ll N^{-\frac{1}{2}} \) can be shown easily by the pigeonhole principle for any distribution of N points on this sphere, the exponent is almost best possible for \( \lambda \) small. However, it is possible (and suggested by a result of Björck; see [3, p. 261, Remark 1]) that \( |p_i - p_j| \gg N^{-\frac{1}{2}} \) for 0 < \( \lambda < 2 \).

It is interesting to observe that spherical distributions of points with maximal spherical distance sums are not necessarily well spaced [6], [8].

For the proof of the Theorem let p1, • • • , pN maximize the sum
where $G$ is a continuous function defined on $[0, 2]$ which has a fourth derivative everywhere in $(0, 2)$. We will choose $G$ later. Rotate $U$ so that $p_1$ coincides with the north pole $(0, 0, 1)$. Let

$$f(p) = \sum_{i=2}^{N} G(|p - p_i|),$$

and let $C$ be a circle on $U$ centered at $p_1$. Clearly $f(p_1)$ must be at least as large as the average value of $f(p)$ on $C$. Choose spherical coordinates $(\theta, \phi)$ so that any point $p = (x, y, z) \in U$ is given by $(\sin \phi \sin \theta, \sin \phi \cos \theta, \cos \phi)$. Thus

$$\frac{1}{2\pi} \int_{0}^{2\pi} f(p) \, d\theta \leq f(0, 0, 1).$$

Next, let $g = g(x, y, z)$ be any function with a convergent power series expansion

$$g(x, y, z) = \sum_{i,j,k} c_{ijk} x^i y^j z^k$$

about $(0, 0, 1)$. For $(x, y, z) \in C \subseteq U$, one has for $\phi$ small,

$$\Delta = \frac{1}{2\pi} \int_{0}^{2\pi} g(x, y, z) \, d\theta - g(0, 0, 1)$$

$$- \sum_{i,j,k} c_{ijk} (\sin \phi)^{i+j} (\cos \phi - 1)^k \frac{1}{2\pi} \int_{0}^{2\pi} \sin^i \theta \cos^j \theta \, d\theta.$$

Now the integral on the right of (7) is zero unless $i$ and $j$ are both even. Hence when $\phi$ is small,

$$\Delta = \frac{1}{4}\phi^2 \left( \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} - 2 \frac{\partial g}{\partial z} \right)_{p_1} + O(\phi^4).$$

In fact (8) is clearly valid under the weaker assumption that all partial derivatives of $g$ of order at most four exist.

Let $q = (x_0, y_0, z_0) \in U$ be fixed. We now set $g(p) = G(u)$, where $u = |p - q|$. Then

$$\frac{\partial g}{\partial x} = \frac{dG}{du} \frac{(x - x_0)}{|p - q|}$$

and

$$\frac{\partial^2 g}{\partial x^2} = \frac{d^2 G}{du^2} \frac{(x - x_0)^2}{|p - q|^2} + \frac{dG}{du} \frac{(y - y_0)^2 + (z - z_0)^2}{|p - q|^3}.$$

Upon evaluating partials at $p = p_1$, we find that the coefficient of $\frac{1}{4}\phi^2$ in
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\[(1 - \frac{u^2}{4}) \frac{d^2 G}{du^2} + (\frac{1}{u} - \frac{3}{4}) \frac{dG}{du},\]

where \(u = |p_1 - q|\). Thus

\[(9) \quad \Delta = \frac{1}{4} \phi^2 \left( \frac{1}{u} \frac{d}{du} \left[ u \left(1 - \frac{u^2}{4}\right) \frac{dG}{du}\right] \right) + O(\phi^4) = a(u)\phi^2 + O(\phi^4).\]

We now replace \(q\) by \(p_i\), where \(2 \leq i \leq N\), and set \(u_i = |p_1 - p_i|\) so (9) becomes

\[(10) \quad \Delta_i = \frac{1}{4} a(u_i)\phi^2 + O(\phi^4).\]

Assume now that the sum (4) is maximal. Let \(s\) be the number of points \(p_i\) which coincide with \(p_1\), and let every point on \(C\) make the small angle \(\phi\) with \((0, 0, 1)\). We apply (5), (7), (9), and (10) with \(g\) replaced by \(f\) (recall (4)), and obtain

\[(11) \quad \Delta = s \cdot 2\pi \sin \phi + \phi^2 \sum_{i=2}^{N} a(u_i) + O(\phi^4) \leq 0.\]

Here the dashed sum is taken over those \(i\) for which \(p_i \neq p_1\). By letting \(\phi \to 0\) we see first that \(s = 0\), so the sum is over all \(i \neq 1\). We then see that the sum is at most zero, i.e.

\[(12) \quad \sum_{i=2}^{N} a(|p_1 - p_i|) \leq 0.\]

If \(G(u) = u^\lambda\), where \(0 < \lambda < 2\), then

\[a(u) = \lambda^2/4u^{2-\lambda} - \lambda(\lambda + 2)u^\lambda/16.\]

Thus for any \(j \neq 1\) we have

\[\lambda^2|p_1 - p_j|^{\lambda-2} \leq \sum_{i=2}^{N} \lambda^2 |p_1 - p_i|^{\lambda-2} \leq \frac{1}{4}\lambda(2 + \lambda) \sum_{i=2}^{N} |p_1 - p_i|^\lambda \leq \frac{1}{4}\lambda(2 + \lambda)2^\lambda \cdot N.\]

Since \(p_1\) was chosen arbitrarily, this completes the proof of the Theorem.

One can now prove many theorems of this type by varying the choice of \(G\). If we take

\[(13) \quad G(u) = \int_{0}^{u} th(t) \ln \left( \frac{u(4 - t^2)^{1/2}}{t(4 - u^2)^{1/2}} \right) dt,\]

then a purely formal calculation gives \(a(u) = 1/4 h(u)\). Thus if the integral
(13) converges for $h(t)$, we have

$$\sum_{i=2}^{N} b(|p_1 - p_i|) \leq 0$$

whenever the sum (3) is maximal. In the case $G(u) = u$, we have $h(t) = t^{-1} - \frac{1}{3}t$. We shall now choose $h(t)$ so that (14) is as strong as possible.

It is not hard to show there is a function $u(t)$ continuous on the closed interval $[0, 2]$ such that (13) converges when $\epsilon > 0$ and

$$h(t) = t^{-2}(\ln(1/t))^{-2}(\ln \ln(1/t))^{-1-\epsilon} - u(t).$$

(In the case $G(u) = u$, the role of $u(t)$ was played by $\frac{1}{3}t$.) It follows (from (14); we suppress the details) that if (3) is maximal, then for $i \neq j$ we have

$$|p_i - p_j| \gg N^{-\frac{1}{2}}(\ln N)^{-1}(\ln \ln N)^{-\frac{1}{2}} - 10\epsilon.$$

Here $G(u)$ behaves like $\epsilon^{-1}(\ln \ln(1/u))^{\epsilon}$ as $u$ approaches 0 from the right. However, we know of no continuous $G$ for which (3) implies $|p_i - p_j| \gg N^{-\frac{1}{2}}$.

REFERENCES


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