DIFFERENTIABILITY OF THE EXPONENTIAL OF A MEMBER OF A NEAR-RING

J. W. NEUBERGER

ABSTRACT. Suppose $S$ is a Banach space and $K$ is the near-ring of all zero preserving Lipschitz transformations from $S$ to $S$. It is shown that all exponentials of members of $K$ have certain differentiability properties. This leads to the fact that no neighborhood of the identity transformation is filled with exponentials of members of $K$.

It is known ([1], [3] or [2]) that if $K$ is a Banach algebra with identity $I$ and $T$ is an element of $K$ such that $|T - I| < 1$, then $T = \text{Exp} A$ for some $A \in K$. Here $\text{Exp} A = \lim_{n \to \infty} (I + (1/n)A)^n$. From this fact it follows easily that the identity component of $K$ is precisely the set of all finite products of exponentials of members of $K$.

One purpose of this note is to show that in a near-ring of zero preserving Lipschitz transformations on a Banach space, it no longer holds that all elements in some neighborhood of $I$ are exponentials. It was shown in [2] that for such near-rings the finite products of exponentials are dense in the identity component. The results of this note indicate that [2] cannot be improved in a certain direction.

Denote by $S$ a real Banach space and by $K$ a near-ring of zero preserving Lipschitz transformations from $S$ to $S$. Precisely, $K$ is a collection of transformations from $S$ to $S$ such that:

1. if each of $T$ and $V \in K$ and $c$ is a number, then $T + V$, $TV$ and $cT \in K$;
2. if $T \in K$, there is a smallest number $|T|$ (called the norm of $T$) such that $\|Tx - Ty\| \leq |T| \|x - y\|$ for all $x, y \in S$;
3. $T0 = 0$;
4. $I$, the identity transformation on $S$, is in $K$;
5. $K$ is complete in the sense that if $T$ is a transformation from $S$ to $S$ such that (2) holds and $\{T_i\}_{i=1}^{\infty}$ is a sequence of elements of $K$ which converges to $T$ uniformly on each bounded subset of $S$, then $T \in K$.

If $A \in K$ then $\text{Exp} A$ denotes the element $L$ of $K$ such that $Lx = \text{Exp} A x$.

Received by the editors December 31, 1973.


Key words and phrases. Identity component, normed near-ring exponential.

1 Supported in part by an NSF grant.

Copyright © 1975, American Mathematical Society

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
differentiability of a near-ring exponential

\lim_{n \to \infty} (l + (1/n)A)^n x \text{ for all } x \in S. \text{ Properties of this exponential function are listed in [2].}

**Theorem 1.** Suppose each of A and B is in K and \((\exp tA)(\exp sB) = (\exp sB)(\exp tA)\) for all numbers \(t\) and \(s\). If \(x \in S\), \(t\) is a number and \(Bx \neq 0\), then \(\exp tA\) is \(G\)-differentiable at \(x\) in the direction \(Bx\) and

\[(D_{Bx} \exp tA)x = (B \exp tA)x \text{ for all numbers } t.\]

**Proof.** From [2, Lemma 0 (iv)], it follows that if \(Q \in K\) and \(x \in S\), then

\[
\|((\exp Q)x - (l + \delta Q)x)\| = \|((\exp Q)x - (x + \delta Qx))\| = o(\delta) \text{ as } \delta \to 0.
\]

Therefore for \(x \in S\) and \(Bx \neq 0\),

\[
\|((\exp tA)(x + \delta Bx) - (\exp tA)x) - \delta(B \exp tA)x\|
\leq \|((\exp tA)(x + \delta Bx) - (\exp tA)(l + \delta B)x)\|
+ \|((\exp tA)(x + \delta B)(\exp tA)x - (\exp tA)(l + \delta B)Bx)\|
\leq \|((\exp tA)(x + \delta B)(\exp tA)x - (\exp tA)(l + \delta B)Bx)\|
+ e^{l^2t^4} \|((\exp \delta B)x - (l + \delta B)x)\|
= o(\delta) \text{ as } \delta \to 0.
\]

So, \(\|((\exp tA)(x + \delta Bx) - (\exp tA)x) - \delta(B \exp tA)x\| = o(\delta) \text{ as } \delta \to 0.\) But this is precisely the statement that \(\exp tA\) is \(G\)-differentiable at \(x\) in the direction \(Bx\) and that \((D_{Bx} \exp tA)x = (B \exp tA)x.\)

**Corollary.** If \(A \in K\), \(t\) is a number and \(x \in S\), then \((D_{Ax} \exp tA)x = (A \exp tA)x.\)

This follows from Theorem 1 since

\[(\exp tA)(\exp sA) = \exp(t + s)A = (\exp sA)(\exp tA)\]

for all numbers \(t\) and \(s.\)

**Theorem 2.** Suppose \(K\) is the near-ring of all zero preserving Lipschitz transformations on \(S.\) Then no neighborhood of \(1\) is filled with exponentials of members of \(K.\)

**Proof.** Suppose \(\epsilon > 0.\) Pick \(y_\epsilon \in S\) such that \(\|y_\epsilon\| = \epsilon\) and define

\[T_\epsilon z = \begin{cases} 
  z & \text{if } \|z - y_\epsilon\| \geq \epsilon, \\
  z - \epsilon \frac{z - y_\epsilon}{\|z - y_\epsilon\|} y_\epsilon & \text{if } \|z - y_\epsilon\| \leq \epsilon.
\end{cases}
\]
It may be verified that \( |T_\varepsilon - I| = \varepsilon \), and if \( x \in S \), \( \|x\| \neq 0 \), \( \alpha \) is a number different from 0 so that \( |\alpha| < \varepsilon /\|x\| \), then
\[
(1/\alpha)[(T_\varepsilon I)(y_\varepsilon + \alpha x) - (T_\varepsilon I)y_\varepsilon] = (|\alpha| /\alpha)\|x\|y_\varepsilon.
\]

Since \( \lim_{\alpha \to 0} (|\alpha| /\alpha)y_\varepsilon \) does not exist, it follows that \( T_\varepsilon - I \) (and hence \( T_\varepsilon \)) is not \( G \)-differentiable in any direction at \( y_\varepsilon \).

Suppose now that \( T_\varepsilon = \exp A \) for some \( A \in K \). Then \( Ay_\varepsilon \neq 0 \), since if it were zero, then
\[
(1 - \varepsilon)y_\varepsilon = T_\varepsilon y_\varepsilon = (\exp A)y_\varepsilon = \lim_{n \to \infty} (I + (1/n)A)^n y_\varepsilon = y_\varepsilon,
\]
a contradiction.

By the Corollary, \( T_\varepsilon = \exp A \) is \( G \)-differentiable in the direction \( Ay_\varepsilon \), a contradiction to the fact established above that \( T_\varepsilon \) is not \( G \)-differentiable at \( y_\varepsilon \) in any direction. Hence the assumption that \( T_\varepsilon = \exp A \) for some \( A \in K \) is false and the theorem is established.

The problem is left open of characterizing those near-rings of zero-preserving Lipschitz transformations on \( S \) which do have a neighborhood of the identity filled with exponentials of members of the near-ring. This is a special case of the following more general problem:

Denote by \( K \) the near-ring of all zero-preserving Lipschitz transformations on a Banach space \( S \). Denote by \( K' \) the identity component of \( K \). Characterize those subgroups of \( K' \) which have a neighborhood of \( I \) which is filled with exponentials of members of \( K \).

A closely related problem is that of finding a necessary and sufficient condition for an element \( T \) of \( K \) (\( |T - I| < 1 \)) to be an exponential of some member of \( K \).

REFERENCES