\textbf{F-PROJECTORS OF FINITE SOLVABLE GROUPS}

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ABSTRACT. \( \mathcal{F} \) denotes a class of finite solvable groups closed under the taking of epimorphic images. For any finite solvable group \( G \) we construct \( \mathcal{F} \)-projectors and prove that they are a single conjugacy class of subgroups of \( G \).

0. Introduction. The concept of a formation was introduced by Gäschutz [3] and showed that in finite solvable groups the Sylow subgroups and Carter subgroups are examples of more general classes called \( \mathcal{F} \)-projectors. If \( \mathcal{F} \) is a saturated formation then every finite solvable group has a single conjugacy class of \( \mathcal{F} \)-projectors. Some results of formation theory have been extended to less restricted classes; see [2], [5], and [7]. In [8] Wielandt localized the concept of a formation by studying \( \mathcal{F}(G) \), the factors of a single group \( G \), and showed that only the properties of the collection \( \mathcal{F} \cap \mathcal{Q}(G) \) are relevant to the existence and conjugacy of \( \mathcal{F} \)-projectors of \( G \). Results analogous to this have been obtained for \( \mathcal{F} \)-normalizers by Prentice [6]. In this note we follow the approach of studying the factors of a single group \( G \) and construct \( \mathcal{F} \)-projectors for any class of factors closed under the taking of epimorphic images within \( G \). The subgroups we obtain coincide with those of Schunk if \( \mathcal{F} \) is a saturated homomorph.

1. \( \mathcal{F} \)-projectors. All groups considered are finite and solvable.

Let \( G \) be a group. We define \( \mathcal{Q}(G) = \{H/H_0:H_0 \trianglelefteq H \trianglelefteq G\} \), the set of factors of \( G \). Let \( \mathcal{J} \) be any subset of \( \mathcal{Q}(G) \).

Definition 1.1. (i) A subgroup \( E \) of \( G \) has the \( \mathcal{F} \)-covering property in \( G \) if whenever \( E \trianglelefteq F \trianglelefteq G \) and \( F/F_0 \in \mathcal{J} \) then \( EF_0 = F \).

(ii) A subgroup \( E \) of \( G \) with the \( \mathcal{F} \)-covering property in \( G \) is an \( \mathcal{F} \)-projector of \( G \) if no proper subgroup of \( E \) has the \( \mathcal{F} \)-covering property in \( E \).
If $\mathcal{F}$ is a saturated formation and $\mathcal{F} = \mathcal{F}^s \cap Q(G)$, the above definition coincides with the usual one.

We will determine conditions on $\mathcal{F}$ that insure that $\mathcal{F}$-projectors of $G$ exist and are a single conjugacy class of subgroups of $G$.

**Definition 1.2.** $\mathcal{F}$ is a homomorph if

1. $A_0 \triangleleft A$, $B_0 \triangleleft B$, $A_0B = A$ and $A_0 \cap B = B_0$, then $A/A_0 \in \mathcal{F}$ if and only if $B/B_0 \in \mathcal{F}$; and
2. $R/R_0 \in \mathcal{F}$ and $R_0 \leq R_1 \triangleleft R$, then $R/R_1 \in \mathcal{F}$.

**Definition 1.3.** $\mathcal{F}$ is normal if and only if whenever $R/S \in \mathcal{F}$ and $g \in G$ then $R^g/S^g \in \mathcal{F}$.

If $\mathcal{F}$ is closed with respect to the taking of epimorphic images within $G$, then Definitions 1.2 and 1.3 are satisfied.

In the rest of the paper, $\mathcal{F}$ will denote a normal homomorph. If $H \leq G$, then $\mathcal{F} \cap Q(H)$ is a normal homomorph of $H$. If $N \triangleleft G$ and we identify the factor $(L/N)/(L_0/N)$ of $G/N$ with the factor $L/L_0$ of $G$, then the set $\mathcal{F} \cap Q(G/N)$ is a normal homomorph of $G/N$. We will refer to $\mathcal{F} \cap Q(H)$-projectors of $H$ and $F \cap Q(G/N)$-projectors of $G/N$ as $\mathcal{F}$-projectors of $G$ and $G/N$, respectively.

**Proposition 1.4.** Let $E$ be an $\mathcal{F}$-projector of $G$.

1. If $E \leq H \leq G$ then $E$ is an $\mathcal{F}$-projector of $H$.
2. If $N \triangleleft G$ then $EN/N$ is an $\mathcal{F}$-projector of $G/N$.

**Proof.** (1) is clear from the definition.

(2) Let $EN/N \leq F/N$ and suppose $N \leq F_0 \triangleleft F$ and $F/F_0 \in \mathcal{F}$. Then since $E \leq F$ and $E$ is an $\mathcal{F}$-projector of $G$, $EF_0 = F$, hence $(EN/N) \cdot (F_0/N) = F/N$. Now suppose $E_1/N$ has the $\mathcal{F}$-covering property in $EN/N$. Suppose $E/E_0 \in \mathcal{F}$. Then $EN/E_0N \in \mathcal{F}$ so $E_1(E_0N) = EN$. Hence since $E_1 \geq N$, $E_1E_0 = EN$ so $E = (E \cap E_1)E_0$ by Dedekind's lemma. Thus $E \cap E_1$ has the $\mathcal{F}$-covering property in $E$ and so $E \cap E_1 = E$. Therefore $EN = E_1$. Thus $EN/N$ is an $\mathcal{F}$-projector of $G/N$.

**Lemma 1.5.** Let $E/N$ be an $\mathcal{F}$-projector of $G/N$ and let $D$ be an $\mathcal{F}$-projector of $E$. Then $D$ is an $\mathcal{F}$-projector of $G$.

**Proof.** By (2) of Proposition 1.4, $DN/N$ is an $\mathcal{F}$-projector of $E/N$. Therefore $DN = E$ since the only subgroup of $E/N$ which has the $\mathcal{F}$-covering property in $E/N$ is $E/N$ itself. Now suppose $D \leq F$ and $F/F_0 \in \mathcal{F}$. Then $E/N = DN/N \leq FN/N$. Hence $E(F_0N) = FN$, i.e., $EF_0 = FN$. Thus $F = (E \cap F)F_0$. Thus $(E \cap F)/(E \cap F_0) \in \mathcal{F}$ and $D \leq E \cap F$. Therefore $D(E \cap F_0)$
= E \cap F, \text{ so } D(E \cap F_0)F_0 = F, \text{ i.e., } DF_0 = F. \text{ Thus } D \text{ is an } \mathcal{J}\text{-projector of } G.

**Theorem 1.6.** If \( \mathcal{J} \) is a normal homomorph of \( G \), then \( G \) has an \( \mathcal{J}\)-projector and the \( \mathcal{J}\)-projectors of \( G \) are a single conjugacy class of subgroups of \( G \).

**Proof.** *Existence.* We proceed by induction on \(|G|\). Let \( N \) be a minimal normal subgroup of \( G \). By induction \( G/N \) has an \( \mathcal{J}\)-projector \( E/N \).

If \( E \vartriangleleft G \) then \( E \) has an \( \mathcal{J}\)-projector \( D \) also by induction. Then \( D \) is an \( \mathcal{J}\)-projector of \( G \) by Lemma 1.5. Thus we may assume that \( G/N \) is its own \( \mathcal{J}\)-projector. Now if \( G \) is its own \( \mathcal{J}\)-projector there is nothing to show, so assume it is not. Then there must exist a subgroup \( E \vartriangleleft G \) with the \( \mathcal{J}\)-covering property in \( G \). Suppose that \( E \vartriangleleft E \) has the \( \mathcal{J}\)-covering property in \( E \). Since \( EN/N \) has the \( \mathcal{J}\)-covering property in \( G/N \), \( E \subseteq N = G/N \) and so \( E \cap N = 1 \). But \( E \subseteq N/N \) has the \( \mathcal{J}\)-covering property in \( E \). Therefore since \( E \vartriangleleft E \), \( E \subseteq E \). Thus \( E \) is an \( \mathcal{J}\)-projector of \( G \).

*Conjugacy.* Clearly, if \( E \) is an \( \mathcal{J}\)-projector of \( G \) so is any conjugate of \( E \), thus we only need to show that any two \( \mathcal{J}\)-projectors of \( G \) are conjugate.

We proceed by induction on \(|G|\). If \( G \) is its own \( \mathcal{J}\)-projector there is nothing to show, so assume \( E, F \) are \( \mathcal{J}\)-projectors of \( G \) and \( E, F \vartriangleleft G \). Let \( N \) be a minimal normal subgroup of \( G \). Then by (2) of Proposition 1.4, \( EN/N \) and \( FN/N \) are \( \mathcal{J}\)-projectors of \( G/N \) so, by induction, \( EN = F^gN \) for some \( g \in G \). Since \( F^g \) is also an \( \mathcal{J}\)-projector, we may assume \( EN = FN \). Now if \( EN \vartriangleleft G \) we may apply induction in \( EN \) to conclude that \( E \) and \( F \) are conjugate because of (1) of Proposition 1.4. Thus we may assume \( EN = FN = G \) for every minimal normal subgroup \( N \) of \( G \). Thus \( E \) and \( F \) are maximal subgroups of \( G \), and \( \text{core}_G E = \text{core}_G F = 1 \) so \( E \) and \( F \) are conjugate by Ore’s theorem [4, p. 165].

2. \( \mathcal{J}\)-crucial chains. In this section the \( \mathcal{J}\)-projectors of \( G \) are characterized by means of the maximal chains joining them to \( G \).

**Definition 2.1.** A complemented chief factor \( H/K \) of \( G \) is \( \mathcal{J}\)-crucial if \( G/H \) is its own \( \mathcal{J}\)-projector but \( G/K \) is not. A complement \( M \) to \( H/K \) is an \( \mathcal{J}\)-crucial maximal subgroup.

**Lemma 2.2.** If \( M \) is an \( \mathcal{J}\)-crucial maximal subgroup of \( G \) which complements the \( \mathcal{J}\)-crucial chief factor \( H/K \), then \( M/K \) is an \( \mathcal{J}\)-projector of \( G/K \).

**Proof.** Without loss of generality, we may assume \( K = 1 \) so \( H \) is a
minimal normal subgroup of $G$. If $M$ does not have the $\mathcal{F}$-covering property in $G$, then $ML \neq G$ for some $L \leq G$ such that $G/L \in \mathcal{F}$. Since $M$ is a maximal subgroup of $G$, it follows that $L \leq M$. Let $E$ be an $\mathcal{F}$-projector of $G$. Since $G/H$ is its own $\mathcal{F}$-projector, $EH = G$. Also $EL = G$ since $G/L \in \mathcal{F}$. Thus $M = M \cap EL = (M \cap E)L$. Let $N_1 = E \cap LH$. Then $N_1 \leq E$ and $H \leq CG(N_1)$ since $LH = L \times H = LN_1$. Thus $N_1 \leq G$. Now $N_1 \not\leq M$ since $L \leq M$ but $LN_1 = LH \not\leq M$. Thus $MN_1 = G$. Now suppose $M/M_0 \leq J$. Then $G/M_0N_1 \in \mathcal{F}$, hence $EM_0N_1 = G$, so $EM_0 = G$. Therefore $M = (M \cap E)M_0$. Therefore $M \cap E$ has the $\mathcal{F}$-covering property in $M$. However $M = G/H$ is its own $\mathcal{F}$-projector. Therefore, $M \leq E$ and $E < G$ since $G$ is not its own $\mathcal{F}$-projector. Hence $M = E$ contrary to assumption. Thus $M$ is an $\mathcal{F}$-projector of $G$.

Lemma 2.3. $G$ is its own $\mathcal{F}$-projector if and only if $G$ possesses no $\mathcal{F}$-crucial maximal subgroups.

Proof. If $G$ has an $\mathcal{F}$-crucial maximal subgroup, then $G$ is not its own $\mathcal{F}$-projector by Lemma 2.2. Let $E$ be an $\mathcal{F}$-projector of $G$ and suppose $E < G$. Then there is a chief factor $H/K$ of $G$ such that $EH = G$ but $EK < G$. Then by (2) of Proposition 1.4, $G/H = EH/H$ is its own $\mathcal{F}$-projector but $G/K > EK/K$ is not. Thus $EK$ is an $\mathcal{F}$-crucial maximal subgroup of $G$.

Definition 2.4. A sequence $G = M_0 > M_1 > \cdots > M_n$ is an $\mathcal{F}$-crucial maximal chain if $M_i$ is an $\mathcal{F}$-crucial maximal subgroup of $M_{i-1}$ for $i \geq 1$ and $M_n$ has no $\mathcal{F}$-crucial maximal subgroups.

Theorem 2.5. (i) If $G = M_0 > \cdots > M_n$ is an $\mathcal{F}$-crucial maximal chain, then $M_n$ is an $\mathcal{F}$-projector of $G$.

(ii) If $E$ is an $\mathcal{F}$-projector of $G$, then there is an $\mathcal{F}$-crucial maximal chain $G = M_0 > M_1 > \cdots > M_n$ such that $E = M_n$.

Proof. (i) We proceed by induction on $n$. If $n = 0$ then $G$ has no $\mathcal{F}$-crucial maximal subgroups so $G$ is its own $\mathcal{F}$-projector and there is nothing to prove. If $n > 0$ let $M_1$ complement the $\mathcal{F}$-crucial chief factor $H/K$. Then an $\mathcal{F}$-projector of $M_1$ is an $\mathcal{F}$-projector of $G$ by Lemma 1.5. However, $M_n$ is an $\mathcal{F}$-projector of $M_1$ by induction so we are done.

(ii) Since $\mathcal{F}$ is normal and $M$ is an $\mathcal{F}$-crucial maximal subgroup of $M_{i-1}$, if $g \in G$, then $M^g$ is an $\mathcal{F}$-crucial maximal subgroup of $M_{i-1}^g$. Since any two $\mathcal{F}$-projectors are conjugate, (ii) now follows from (i).

3. Examples. In this section $\mathcal{X}$ will denote a class of groups closed under the taking of epimorphic images. If $G$ is a group then $G$ has $\mathcal{X} \cap Q(G)$-
projectors which we will refer to as $\mathcal{X}$-projectors. In general, the $\mathcal{X}$-projectors of $G$ are not simply the $\mathcal{Y}$-projectors of $G$ for some saturated formation $\mathcal{Y}$ as the next theorem shows.

**Theorem 3.1.** Let $\mathcal{Y}$ be a saturated formation such that $\mathcal{X} \leq \mathcal{Y}$. For every group $G$, the $\mathcal{X}$-projectors of $G$ coincide with the $\mathcal{Y}$-projectors of $G$ if and only if $\mathcal{Y} = \phi R_0 \mathcal{X}$.

**Proof.** (i) Suppose that for every group $G$ the $\mathcal{X}$-projectors of $G$ coincide with the $\mathcal{Y}$-projectors of $G$. Since $\mathcal{Y}$ is saturated, $\phi R_0 \mathcal{X} \leq \mathcal{Y}$. Let $G \in \mathcal{Y}$. $G$ is its own $\mathcal{X}$-projector by assumption, hence $G$ has no $\mathcal{X}$-crucial maximal subgroups. Therefore $G/\text{core}_G M$ is its own $\mathcal{X}$-projector for every maximal subgroup $M$ of $G$. Since $M/\text{core}_G M$ does not have the $\mathcal{X}$-covering property in $G$, there exists $N/\text{core}_G M < G/\text{core}_G M$ such that $G/N \in \mathcal{X}$ but $MN \not\in G$. However, $G/\text{core}_G M$ is monolithic so $N = \text{core}_G M$. Thus $G/\text{core}_G M \in \mathcal{X}$ for every maximal subgroup $M$ of $G$. Let $C = \bigcap \{\text{core}_G M | M \text{ is a maximal subgroup of } G\}$. Then $G/C \in R_0 \mathcal{X}$ and $C = \phi(G)$, so $G \in \phi R_0 \mathcal{X}$.

(ii) Suppose now that $\mathcal{Y} = \phi R_0 \mathcal{X}$ is a saturated formation. Let $G$ be a group; we prove by induction on $|G|$ that the $\mathcal{X}$-projectors of $G$ coincide with the $\mathcal{Y}$-projectors of $G$. If $G$ is its own $\mathcal{X}$-projector, then $G$ has no $\mathcal{X}$-crucial maximal subgroups, and the proof of (i) shows that $G \in \phi R_0 \mathcal{X} = \mathcal{Y}$, so $G$ is its own $\mathcal{Y}$-projector. If $G$ is not its own $\mathcal{X}$-projector then $G$ possesses an $\mathcal{X}$-crucial maximal subgroup $M$ which complements an $\mathcal{X}$-crucial chief factor $H/K$. Then $G/H$ is its own $\mathcal{X}$-projector, so $G/H \in \mathcal{Y}$. Suppose $G/K \in \mathcal{Y}$. Then since $\mathcal{Y}$ is a saturated formation, $G/\text{core}_G M \in \mathcal{Y}$. But $G/\text{core}_G M$ is $\phi$-free and monolithic, so $G/\text{core}_G M \in \mathcal{X}$, a contradiction to $M$ being $\mathcal{X}$-crucial. Thus $G/K \not\in \mathcal{Y}$. Hence $H/K$ and, therefore, $M$ is $\mathcal{Y}$-crucial. Thus the $\mathcal{X}$- and $\mathcal{Y}$-projectors of $G$ coincide with the $\mathcal{X}$- and $\mathcal{Y}$-projectors of $M$, respectively, but the latter coincide by induction so we are done.

The question of when $\phi R_0 \mathcal{X}$ is a saturated formation has been answered by Cossey and McDonald [1].

**Theorem 3.2** (see [1, Lemma 2.3]). Let $\mathcal{X}$ be a class of groups. $\mathcal{Y} = \phi R_0 \mathcal{X}$ is a saturated formation if and only if $\mathcal{X}$ contains every $\phi$-free monolithic group in $\mathcal{Y}$.

**Example 3.3.** If $\mathcal{X}$ is the class of all cyclic groups of square free order, then $\phi R_0 \mathcal{X} = \mathcal{N}$ the class of all nilpotent groups. If $\mathcal{X} = \{1, C_p\}$ then $\phi R_0 \mathcal{X}$ is the class of all $p$-groups.
Example 3.4. Let $\mathcal{X} = \{1, C_2, S_3\}$; then $\mathcal{Y} = \phi R_0 \mathcal{X}$ is not a saturated formation. If $\mathcal{Y}$ were a saturated formation, then $C_3 \in \mathcal{Y}$, but $C_3$ is $\phi$-free and monolithic, so it would follow, by Theorem 3.2, that $C_3 \in \mathcal{X}$, which is not true.

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