ABSTRACT. Suppose that $M$ is a time oriented, future 1-connected, timelike and null geodesically complete Lorentzian manifold. Previously, we have shown the exponential map at any point of such a manifold embeds the future cone into $M$ when $M$ has nonpositive spacetime curvatures. Here we want to demonstrate that under the same hypotheses, $M$ is homeomorphic to the product of the real line with a Cauchy hypersurface.

Let us recall briefly the main points in [F]. The basic object of study is a Lorentzian $n$-manifold $M$ (signature $2-n$), which we suppose time orientable. We tacitly assume then that $M$ is time oriented. The manifold $M$ is called future 1-connected iff any two future-timelike (smooth) curves from $p$ to $q$ are homotopic through future-timelike curves with endpoints $p$ and $q$. The spacetime curvatures of $M$ are the sectional curvatures of planes spanned by a timelike and a spacelike vector. We now state Proposition 2.1 of [F]: Let $M$ be a time oriented Lorentzian manifold with nonpositive spacetime curvatures; then the exponential map at any point of $M$ has maximal rank on the (closed) future cone. This proposition is very useful in proving the main theorem of [F], which is stated here in the first paragraph.

Briefly our intention is to introduce the notion of a globally hyperbolic manifold, much the same as Leray did in [L], and prove that our manifolds are globally hyperbolic. Our theorem then follows from a result of Geroch, that globally hyperbolic spaces are homeomorphic to a product of the real line with a Cauchy hypersurface [G].

The first problem is to extend the ideas of timelike or null—defined only for smooth curves—to continuous curves. We use essentially the definition of Hawking and Ellis [HE] for nonspacelike curves. A continuous curve $c$ mapping an interval of real numbers $I$ into $M$ is called a nonspacelike curve iff for any $t$ in $I$ there is an $\eta > 0$ and a normal neighborhood $U$ of
$c(t)$ such that for $t + \eta > s > t$ there is a future-timelike or -null curve joining $c(s)$ to $c(t)$ in $U$, and if $t - \eta < s < t$, there is a past-timelike or -null curve joining $c(s)$ to $c(t)$ in $U$. Intuitively nonspacelike curves are continuous timelike or null curves. A more thorough discussion can be found in [G] or [HE].

Let $C(p, q)$ denote the set of all equivalence classes of nonspacelike curves from $p$ to $q$ under the relation of reparameterization by continuous monotonic maps. Provide $C(p, q)$ with the compact open topology and observe that

$$C(p, q) = C^+(p, q) \cup \Omega_0(p, q),$$

where $C^+(p, q)$ is the space of timelike curves from $p$ to $q$ and $\Omega_0(p, q)$ is the space of unbroken null geodesics from $p$ to $q$ without conjugate points. This follows from [HE, §4.5]. Following Leray, a time oriented manifold $M$ is said to be globally hyperbolic iff $C(p, q)$ is empty or compact for all $p$ and $q$ in $M$. Geroch gives a geometric way of looking at the convergence of curves in $C(p, q)$ when there are no closed nonspacelike curves, compare [G].

**Theorem.** Let $M$ be a future 1-connected manifold which is timelike and null geodesically complete. Further suppose that the spacetime curvatures of $M$ are nonpositive. Then $M$ is globally hyperbolic.

**Proof.** Suppose that $C(p, q)$ is nonempty. If $C^+(p, q)$ is empty, then $C(p, q) = \Omega_0(p, q)$. Let $N_0$ be the set of all null vectors $u$ such that $c(t) = \exp(tu)$ is in $\Omega_0(p, q)$. Clearly $N_0$ is a discrete set because $\exp$ has maximal rank at each $u$ in $N_0$ by Proposition 2.1 of [F]. Now if $N_0$ were an unbounded set, we could find a sequence $(u_n)$ from $N_0$ such that $u_n \to \infty$. So, given neighborhoods $U_n$ of radius $1/n$ around $q$, we can find, by continuity, neighborhoods $V_n$ of $u_n$ such that $\exp(V_n) \subset U_n$. If we take a sequence of timelike vectors $v_n$ in $V_n$, then it follows that $\exp(v_n) = q_n \to q$. In addition, $v_n$ can be chosen arbitrarily close to $u_n$ and so $v_n \to \infty$ as well. Again from Proposition 2.1 of [F], it follows that $\exp$ is of maximal rank at $u_1$, $\exp(u_1) = q$, and so there are compact neighborhoods $U$ of $q$ and $V$ of $u_1$ such that $\exp$ maps $V$ diffeomorphically onto $U$. Further, there is a positive integer $n_0$ for which $n \geq n_0$ implies $q_n$ is in $U$. From the main theorem of [F], $\exp$ has an inverse on the set of timelike vectors, and the restriction of this inverse to the image of timelike vectors in $V$ must agree with the inverse of the map $\exp_V$ (restriction to $V$) on the image of timelike vectors.
in $V$. Hence

$$\exp^{-1}(q_n) = \exp^{-1}(\exp(v_n)) = v_n$$

for $n \geq n_0$; so the $v_n$ are in $V$, contradicting the fact that $v_n \to \infty$. As a result $u_n$ does not go to infinity, and thus the set $N_0$ is bounded and so finite. In this case $C(p, q)$ is compact. If $C^+(p, q)$ were nonempty, $q = \exp(u)$ for some timelike $u$ (again by the main theorem of [F]). First we want to prove that $\exp: C(0, u) \to C^+(p, q)$ is onto, where $C(0, u)$ is the set of nonspacelike curves in the tangentspace from 0 to $u$. Thus for $c = \lim c_n$, $c_n$ in $C^+(p, q)$, the curves $a_n = \exp^{-1}c_n$ are timelike curves in the tangentspace, by a similar argument as in the proof of the main theorem of [F]. It follows from the compactness of $C(0, u)$ that $a_n \to a$, possibly passing to a subsequence, and since $\exp$ is defined on the closed cone, $\exp(a)$ makes sense. Moreover the map $\exp: C(0, u) \to C(p, q)$ is continuous in the compact open topology so $\exp(a) = c$. Again $\Omega_0(p, q)$ is the image of a finite set, so $C(p, q)$ is the continuous image of a compact set and, hence, compact.

Finally, we state Geroch's result on globally hyperbolic manifolds. First, a subset $S$ of a Lorentzian manifold is called achronal iff no $p$ and $q$ in $S$ can be joined by a timelike curve. Now define $D^+(S)$ (respectively $D^-(S)$) to be the set of points $p$ such that every past (respectively future) directed timelike curve from $p$ without a past (respectively future) endpoint intersects $S$. An achronal subset $S$ of $M$ is called a Cauchy hypersurface iff $D^+(S) \cup D^-(S) = M$.

**Theorem [G].** $M$ is globally hyperbolic iff $M$ contains a Cauchy hypersurface $S$, in which case $M$ is homeomorphic to the product of the real line with $S$.

**Theorem.** Let $M$ be a future 1-connected manifold which is timelike and null geodesically complete. Further suppose that the spacetime curvatures of $M$ are nonpositive. Then $M$ is homeomorphic to the product of the real line with a Cauchy hypersurface.

In conclusion, let us give an example of a Lorentzian manifold that is simply connected but not future 1-connected. This example is due to R. P. Geroch. Consider ordinary Minkowski three-space with coordinates $x, y,$ and $t$. Let $U$ be the open set $|t| < 1$ with the positive $x$-axis removed as well as the interval from $-2$ to $2$ on the $y$-axis. The union of the removed
sets is an infinite $T$-shaped figure. The set $U$ is clearly simply connected. Choose points $p$ and $q$ above and below the $x$-axis, respectively, and join them by timelike curves straddling the $x$-axis. These two curves cannot be homotoped by timelike curves, since you would have to go around the part of the $y$-axis that has been excluded, which is impossible without introducing spacelike curves. Avez has considered a similar notion of timelike homotopy in [A] as has J. W. Smith in [S].

Finally it would be interesting to prove this theorem with future 1-connected replaced by simply connected.

REFERENCES


DEPARTMENT OF MATHEMATICS, OREGON STATE UNIVERSITY, CORVALLIS, OREGON 97331

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use