A NOTE ON A COROLLARY OF SARD'S THEOREM

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ABSTRACT. A corollary of Sard's theorem is the following:

Corollary. Let \( f: K \to \mathbb{R}^n \) be a smooth (i.e. \( f \in C^k, k \geq 1 \)) map defined on a compact subset \( K \) of \( \mathbb{R}^n \). Let \( C = \{ y | f^{-1}(y) \text{ is infinite} \} \). Then the Lebesgue measure of \( C \) is zero.

The purpose of this note is to show that a similar version of this theorem holds for Lipschitz functions.

Theorem. Let \( f: K \to \mathbb{R}^n \) with \( f \) Lipschitz, \( K \) a measurable subset of \( \mathbb{R}^n \) and \( m(K) \) the Lebesgue measure of \( K \), less than \( \infty \). Let \( C = \{ y | f^{-1}(y) \text{ is infinite} \} \). Then \( m(C) = 0 \).

The proof of the theorem will require two lemmas.

**Lemma 1.** Let \( f: X \to \mathbb{R}^m \) with \( X \) a compact subset of \( \mathbb{R}^n \) and \( f \) a continuous function. Then there exists \( K' \subset K \) with \( f(K') = f(K) \), \( f|_{K'} \) one to one and \( K' \) a Borel set.

Proof. Let \( x_i \) denote the \( i \)th coordinate of \( x \). Define for \( j = 1, 2, 3, \ldots \),

\[ A_j = \{ x \in X : \text{there exists } y \in X \text{ with } y_j \leq x_j - 1/j \text{ and } f(y) = f(x) \} \]

or\[ A_j = \{ x \in X : \text{there exists } y \in X \text{ with } y_j = x_j, y_j \leq x_j - 1/j \text{ and } f(y) = f(x) \} \]

or

\[ A_j = \{ x \in X : \text{there exists } y \in X \text{ with } y_j = x_j, \ldots, y_{n-1} = x_{n-1}, y_n \leq x_n - 1/j \text{ and } f(y) = f(x) \} \].

Since \( A_j \) is compact for each \( j \), \( K' = K - \bigcup_j A_j \) is Borel. If \( x \in f(K) \) observe that \( y \) defined by

\[ y_1 = \inf \{ z_1 | f(z) = f(x) \}, \ldots, y_n = \inf \{ z_n | z_1 = y_1, \ldots, z_{n-1} = y_{n-1}, f(z) = f(x) \} \]

is not contained in \( A_j \) for any \( j \) and also \( f(y) = f(x) \). Thus \( f(K') = f(K) \). Clearly \( f|_{K'} \) is one to one.

**Lemma 2.** Let \( f: K \to \mathbb{R}^m \) be a continuous map from \( K \) a measurable subset of \( \mathbb{R}^n \). Then there exist \( K', K'' \) with \( K'' \subset K' \subset K \), \( K'' \) and \( K' \) Borel sets, \( f|_{K''} \) one to one, \( m(K - K') = 0 \) and \( f(K'') = f(K') \).

Proof. There exists an \( F_{\sigma} \) set \( K' \), with \( K' \subset K \) and \( m(K - K') = 0 \), and we can write \( K' = \bigcup_n K_n \) where \( K_n \) is compact for each \( n \). By Lemma 1...
there are Borel sets \( K'_n \subseteq K_n \) with \( f(K'_n) = f(K_n) \) and \( f|_{K'_n} \) one to one for each \( n \). Define recursively \( D_1 = K'_1, \ldots, D_k = D_{k-1} \cup (K'_k - f^{-1}(f(D_{k-1}))) \).

Then \( K'' = \bigcup_k D_k \) is Borel, \( f(K'') = f(K') \) and \( f|_{K''} \) is one to one.

**Proof of Theorem.** Suppose \( m(C) = a > 0 \). Let \( A = f^{-1}(C) \) and find by Lemma 2 \( N'' \subseteq N' \subseteq A \) with \( N'' \) and \( N' \) Borel, \( m(A_1 - N'_1) = 0 \), \( f|_{N''} \) one to one and \( f(N''_1) = f(N'_1) \). Since \( f \) is Lipschitz, \( m(f(A_1 - N'_1)) = 0 \). Therefore \( m(f(N''_1)) = a \). Now \( f(A_1 - N''_1) = C \) by the definition of \( C \). Thus for \( k = 2, 3, \ldots \) by letting \( A_k = A_{k-1} - N''_{k-1} \) we can repeat the above argument to find \( N''_k \subseteq N'_k \subseteq A_k \) with \( f|_{N''_k} \) one to one, \( f(N''_k) = f(N'_k) \) and \( m(f(N''_k)) = a \).

Suppose that \( L \) is the Lipschitz constant for \( f \); then \( m(N''_k) \geq a/L \) for each \( k \). But since \( N''_k \cap N'_k = \emptyset \) for \( k = k' \), this implies \( m(K) = \infty \), a contradiction.

**Remarks.** The requirement that \( f \) be Lipschitz in the theorem cannot be weakened to a requirement of continuity. To show this let \( f \) be the first coordinate of a continuous map of the unit interval \( I \) onto the unit square \( I^2 \) (i.e. a space-filling curve). Then \( f^{-1}(y) \) is infinite for each \( y \in I \).

On the other hand, for \( n = 1 \) absolute continuity will suffice as the following argument shows. Suppose \( f \) is a real valued a.c. function on \( I \). Then \( f \) is differentiable except on a set \( A_1 \) of measure zero. \( m(f(A_1)) = 0 \) since \( f \) is a.c. Let \( A_2 = \{ x | f'(x) = 0 \} \) and let \( \eta \) be an arbitrary positive number. For each \( x \in A_2 \) there are arbitrarily small intervals containing \( x \) such that \( |f(y) - f(x)| \leq |y - x| \) when \( y \) is in the intervals. Thus the collection \( \mathcal{F} \) of all such intervals with \( x \) varying over \( A_2 \) is a Vitale covering of \( A_2 \), so there is a disjoint collection \( \{ I_n \} \subseteq \mathcal{F} \) with \( m(A_2 - \bigcup_n I_n) = 0 \). But then \( m(f(A_2)) \leq m \left( f \left( A_2 - \bigcup_n I_n \right) \right) + m \left( f \left( \bigcup_n I_n \right) \right) \leq 0 + 2\eta \cdot \sum_n \text{length}(I_n) \leq 2\eta. \)

Since \( \eta \) is arbitrary, \( m(f(A_2)) = 0 \). Finally, if \( y \in f(I) - f(A_1) - f(A_2) \) then \( f^{-1}(y) \) must consist of isolated points and hence be finite. Thus \( m|_{f^{-1}(y)} \) is infinite \} = 0.

**REFERENCES**


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