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THE DOMAIN RANK OF A SURFACE IS COUNTABLE

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ABSTRACT. In this paper it is shown that any surface has at most countably infinite domain rank.

1. **Introduction.** For the convenience of the reader we first recall some of the definitions found in [1], [2], and [3]. A *surface* M is a connected, separable, metrizable 2-manifold and a domain D of a surface M is an open connected subset of M. If M is a surface, a set of surfaces G^* is called a *set of generating domains for* M if given any proper domain D of the surface M, there is an element G_D of G^* such that $D = \bigcup_{k=1}^{\infty} D_k$ where for all k, D_k ia a domain of M, $D_k \subset D_{k+1}$, and D_k is homeomorphic to G_D (in such case we say that G_D generates D or that D is an open monotone union of G_D). The *domain rank of* M, denoted by $DR(M)$, is defined by

$$DR(M) = \text{g.l.b.}\{|G^*| : G^* \text{ is a set of generating domains for } M\},$$

where $|G^*|$ denotes the cardinality of G^*.

The study of the domain rank of a surface is naturally motivated by the fact established in [1] that any domain D of the plane R^2 is generated by $R^2 - \{(0, n) | n \text{ is a positive integer}\}$. This paper completes the study of this problem which has been previously considered in [1], [2], and [3].

2. **Proof of the Theorem.** Let M be a noncompact surface, bd $M \neq \emptyset$. Let $s(M)$ be the number of components of bd M which are 1-spheres. If $s(M) = 0$, let $M_c = M$; otherwise let $s(M) = s$, $1 \leq s \leq \infty$, and let $\{S_j\}_{j=1}^{s}$ be the collection of boundary components which are homeomorphic to S^1. Let $Q = +_{j=1}^{s} D_j$ be a disjoint union of 2-cells, f: bd $Q \to M$ be a continuous function which maps bd D_j homeomorphically onto S_j, and M_c be the surface defined by $M_c = Q \cup_f M$. We will identify M, S_j, and D_j with their embedded images under the identification p: $Q + M \to M_c$ and thus $M = M_c - (\bigcup_{j=1}^{s} \text{int } D_j)$. Let $N = \text{bd } M_c \times [0, 1) \cup_g M_c$ where $g(x, 0) = x$ for

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all $x \in \text{bd } M_c$. Again, identify $\text{bd } M_c \times [0, 1)$ and all subsets of M_c with their embedded images under the identification q: $\text{bd } M_c \times [0, 1) + M_c \to N$. As in [3], let (X, Y, Z) denote the ideal boundary of the open surface N. Thus in view of Theorem 1.1 of [3], we may assume that N is obtained from S^2 by removing X and then removing the interiors of a finite or infinite sequence $\{C_k\}$ of pairwise disjoint 2-cells and properly identifying the boundaries of these 2-cells. Furthermore, it is clear that we may assume that each of these 2-cells does not meet $(\text{bd } M_c \times [0, 1) \cup (\bigcup_{j=1}^{s} D_j))$.

Let $W = \text{bd } M_c \times (0, 1)$ and $L_c = S^2 - (X \cup W)$. Then we can obtain M_c from L_c by removing the interiors of the 2-cells C_k and properly identifying the boundaries.

Now consider a point $b \in$ ideal boundary X of N. We will say b is *peripheral to* $M \Leftrightarrow$ there exists some component K of $\text{bd } M_c = \text{bd } L_c$ such that the closure of K in S^2 is $K \cup \{b\}$. If b is peripheral to M, then a *boundary leaf of* M *at* b is any closed set of the form $K \cup \{b\}$ where K is a boundary component of M_c.

We now wish to construct a domain D which generates M. Let p denote the genus of M, $0 \leq p \leq \infty$; q the class of N = class of int M, $1 \leq q \leq 4$; $s = s(M)$ the number of components of bd M which are 1-spheres, $0 \leq s \leq \infty$; and t the number of points $b \in$ ideal boundary of N which are peripheral to M, $0 \leq t \leq \infty$. Let F be the closed subset of S^2 defined by

$$F = X \cup \text{bd } M_c \times [0, 1) = X \cup W \cup \text{bd } L_c.$$

Assume that the sequence of disks $\Gamma_1 = \{C_k\}_{k=1}^{r}$, $0 \leq r \leq \infty$, is chosen as in the remark following Theorem 1.1 of [3]. If p is finite, then r is finite and we set $\Gamma_1' = \Gamma_1$. If p is infinite, then as in the proof of Theorem 2.1 of [3] we may choose an appropriate subsequence $\Gamma_1' = \{C_k'\}_{k=1}^{\infty}$ of Γ_1 which converges to a point $x_1 \in X$. Furthermore, we may assume that this subsequence is chosen such that for all cases, $0 \leq p \leq \infty$, we can construct a 2-cell B_1 in S^2 such that $B_1 \cap F = x_1$, $B_1 \cap (\bigcup_{j=1}^{s} D_j) = \emptyset$, $C_k' \subset \text{int } B_1$ for all k, and $C_k \cap B_1 \neq \emptyset \Leftrightarrow C_k \in \Gamma_1'$.

Now let $\Gamma_2 = \{D_j\}_{j=1}^{s}$ be the collection of 2-cells corresponding to the 1-sphere boundary components of M. If s is finite, let $\Gamma_2' = \Gamma_2$. If $s = \infty$, then we can find a subsequence $\Gamma_2' = \{D_j'\}_{j=1}^{\infty}$ which converges to some unique point $x_2 \in X$. Furthermore, we may assume this subsequence is chosen so that for all cases, $0 \leq s \leq \infty$, we can construct a 2-cell B_2 in S_2 such that $B_2 \cap F = x_2 \in X$, $B_2 \cap (\bigcup_{k=1}^{r} C_k) = \emptyset$, $D_j' \subset \text{int } B_2$ for all j,

$D_j \cap B_2 \neq \emptyset \Leftrightarrow D_j \in \Gamma_2'$, and $B_1 \cap B_2 \neq \emptyset \Rightarrow$ that $x_1 = x_2$ and $B_1 \cap B_2 = \{x_1\} = \{x_2\}$.

We now consider the points $b \in X$ which are peripheral to M. Observe that for each component K_j of bd M_c = bd L_c, the closure of $[0, 1) \times K_j = [0, 1) \times K_j \cup \{b_j\}$ where $b_j \in X$. If each set F_j = closure of $[0, 1) \times K_j$ in S_2 is shrunk to the point b_j, we obtain S_2 and X represents the ideal boundary of int M_c. Let

$$T = [0, 1) \times \text{bd } M_c \cup \left(\bigcup_{j=1}^{s} D_j\right) \cup \left(\bigcup_{k=1}^{r} C_k\right).$$

If the number t of peripheral points is infinite, let $\Omega = \{b_i\}_{i=1}^{\infty}$ be a sequence of peripheral points such that $b_i \neq x_1$ or x_2 for all i and Ω has a unique limit point $x_3 \in X$. Then we can construct a sequence $\Gamma_3 = \{E_i\}_{i=1}^{\infty}$ of pairwise disjoint 2-cells with the following properties:

(a) $b_i \in \text{int } E_i$;

(b) $E_i \cap B_v = \emptyset$ for all i and $v = 1$ and 2;

(c) there is a closed 2-annulus $A_i \subset E_i$ with bd $E_i \subset A_i$ such that $A_i \cap T = \alpha_i$, α_i a closed subset of some boundary leaf at b_i, α_i homeomorphic to $[0, 1]$ and $b_i \notin \alpha_i$; and

(d) $\text{cl}(\bigcup_{i=1}^{\infty} E_i) = (\bigcup_{i=1}^{\infty} E_i) \cup \{x_3\}$, $x_3 \in X$.

For each i, let $F_i = \text{cl}(E_i - A_i)$. We may further assume that Γ_3 is chosen so that we can construct a 2-cell B_3 of S^2 such that

(e) $E_i \subset \text{int } B_3$ for all i;

(f) $(B_3 - (\bigcup_{i=1}^{\infty} F_i)) \cap X = \{x_3\}$;

(g) $(B_3 - (\bigcup_{i=1}^{\infty} F_i)) \cap T = \emptyset$; and

(h) $B_3 \cap B_v \neq \emptyset$ for $v = 1$ or $2 \Rightarrow x_3 = x_v$ and $B_3 \cap B_v = \{x_v\}$.

If t is finite, and $t' \geq 1$, where t' is the cardinality of the set of peripheral points $-\{x_1, x_2\}$, we can similarly construct 2-cells $\{E_i\}_{i=1}^{t'}$, with properties (a)–(c) and a 2-cell B_3 with properties analogous to (e)–(h) above where x_3 is some point of X. If $t' = 0$, construct B_3 satisfying appropriate properties similar to (f) and (h).

Finally, construct a 2-cell B_4 such that $B_4 \cap X = \{x_1\}$, $B_4 \cap T = \emptyset$, and $(B_4 - \{x_1\}) \cap B_v = \emptyset$, $v = 1, 2, 3$; and let $\{H_n\}_{n=1}^{\infty}$ be a disjoint sequence of 2-cells contained in int B_4 which converge to x_1.

We will say a 2-cell B contained in a 2-cell C is 1-cell proper in $C \Leftrightarrow B \cap \text{bd } C$ is a 1-cell. Let $Z = T \cup (\bigcup_{v=1}^{4} B_v)$. Then we can find a set Q of disjoint 2-cells $\{Q_1, Q_2, Q_3\}$ with $Q_i = Q_j$ if $x_i = x_j$ such that

$x_i \in \operatorname{int} Q_i$ for all i and such that there is a 2-cell $P_i \subset \operatorname{int} Q_i$ with $x_i \in \operatorname{int} P_i$ and $(Q_i - \operatorname{int} P_i) \cap Z = \emptyset$. Furthermore, we may assume the following depending on the relation between x_1, x_2, and x_3:

Case 1. *x_1, x_2, and x_3 are all distinct.* B_v is 1-cell proper in P_v, $v = 1, 2, 3$, and B_4 is 1-cell proper in P_1.

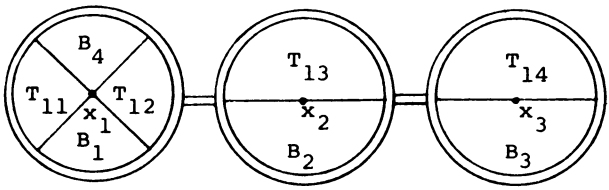
Case 2. $x_1 = x_2$. B_3 is 1-cell proper in P_3 and B_v is 1-cell proper in P_1, $v = 1, 2$, and 4.

Case 3. $x_2 = x_3$. B_1 and B_4 are 1-cell proper in P_1 and B_2 and B_3 are 1-cell proper in P_2.

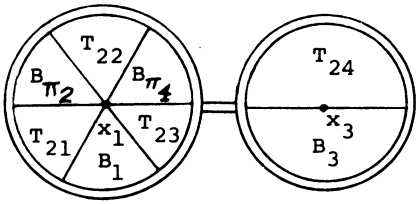
Case 4. $x_1 = x_2 = x_3$. B_v is 1-cell proper in P_1 for $1 \leq v \leq 4$.

By joining (if necessary) the 2-cells of the set Q together by 2-cells lying in $S^2 - (T \cup X)$ we obtain a 2-cell K which, depending on the case, has the following form:

Case 1

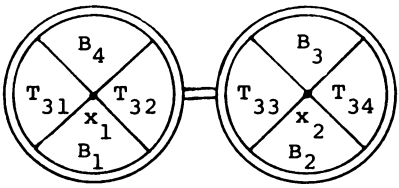


Case 2

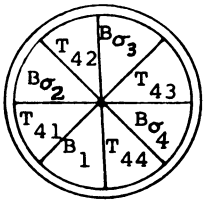


π a permutation of $\{2, 4\}$

Case 3



Case 4



σ a permutation of $\{2, 3, 4\}$

We now consider surfaces S of genus 0 contained in the surface L_c which take one of the following forms:

(i) $T_{ij} \cap L_c$, $1 \leq i, j \leq 4$;

(ii) $(S^2 - K) \cap L_c$; or

(iii) $E_i \cap L_c$, $0 < i < M(i)$, where $M(i)$ depends on M.

Note that for any of the above surfaces S, $C_k \cap S \neq \varnothing \Leftrightarrow C_k \subset \operatorname{int} S$ and $D_j \cap S \neq \varnothing \Leftrightarrow D_j \subset \operatorname{int} S$ and that the union of all C_k and D_j which meet a fixed S nonvoidly is closed in S. Also $\operatorname{bd} T_{ij}$, $\operatorname{bd}(S^2 - K)$, and $\operatorname{bd} E_i$ do not meet $\operatorname{bd} L_c$.

We now describe how to obtain a domain D_c which generates L_c and "respects the collection of 2-cells Γ_1 and Γ_2". A domain D which generates M is then easily obtained from D_c.

To obtain D_c, it is necessary to replace each surface S of the type mentioned by an appropriate surface of genus 0. Henceforth we will say that a surface $Q \subset S^2$ meets

$$R = \left(\bigcup_{j=1}^{s} D_j\right) \cup \left(\bigcup_{k=1}^{r} C_k\right)$$

finitely $\Leftrightarrow$ at most a finite number of elements of $\{D_j\}$, $\{C_k\}$ intersect Q nonvoidly and in such case are contained in $\operatorname{int} Q$. Note that any surface S of the type mentioned above has one of the following properties:

(1) $S = T_{ij} \cap L_c$ and T_{ij} contains a boundary leaf at x_v, $1 \leq v \leq 3$, $x_v \in \operatorname{bd} T_{ij}$. Let X_{ij} be a closed subset of S such that $X_{ij} \equiv R \times [0, 1]$, closure of X_{ij} in S^2 is $X_{ij} \cup \{x_v\}$, $\operatorname{bd} T_{ij} - \{x_v\}$ corresponds to $R \times \{0\} = X_{ij0}$ and $X_{ij} \cap T = \varnothing$. Let X_{ij1} denote the boundary component of X_{ij} corresponding to $R \times \{1\}$. Then we can find a sequence of compact surfaces $\{L_{ijt}\}_{t=0}^{\infty}$ of genus 0 with the following properties:

(a) $L_{ijt} \cap X_{ij} = L_{ijt} \cap X_{ij1}$ is a 1-cell;

(b) $L_{ijt} \cap \operatorname{bd} S$ is a finite nonempty collection of 1-cells at least one of which lies in the same component of $\operatorname{bd} L_{ijt}$ as $L_{ijt} \cap X_{ij1}$;

(c) L_{ijt} meets R finitely;

(d) $L_{ijt} \subset i_A L_{ij(t+1)}$ = point set interior of $L_{ij(t+1)}$ relative to $A = (S - X_{ij}) \cup X_{ij1}$;

(e) L_0 is a 2-cell, L_0 does not meet R, and $L_0 \cap \operatorname{bd} S$ is a 1-cell; and

(f) $$S = X_{ij} \cup \left(\bigcup_{t=0}^{\infty} i_A L_{ijt}\right) = \bigcup_{t=0}^{\infty} i_S(X_{ij} \cup L_{ijt}).$$

(2) $S = T_{ij} \cap L_c$ and T_{ij} contains no boundary leaf at x_v, $1 \leq v \leq 3$,

$x_v \in \text{bd } T_{ij}$. Choose X_{ij} as in (1). Then we can find a sequence of compact surfaces $\{L_{ijt}\}_{t=0}^{\infty}$ of genus 0 with properties (a), (c), (d), and (f) as in (1) but with (b) and (e) as follows:

(b) $L_{ijt} \cap \text{bd } S$ is a finite collection of 1-cells, none of which lie in the same component of bd L_{ijt} as $L_{ijt} \cap X_{ij1}$; and

(e) L_0 is a 2-cell, L_0 does not meet R, and $L_0 \cap \text{bd } S = \varnothing$.

(3) $S = (S^2 - K) \cap L_c$ and S is a 2-cell. Note that we may always assume that this is the case since if not, we can move any 2-cell contained in $\text{int}(S^2 - K)$ into $\text{int}(T_{i1})$ moving only points inside an appropriate 2-cell and thus obtain the desired situation.

(4) $S = E_i \cap L_c$ and E_i contains a boundary leaf at b_i, $b_i \in \text{int } F_i$. Then we can find a sequence of compact surfaces $\{N_{it}\}_{t=0}^{\infty}$ of genus 0 with the following properties:

(a) $N_{it} \cap \text{bd } E_i = \text{bd } E_i$ for all t;

(b) Let $J_i = \bigcup\{M | M \text{ is a leaf at } b_i\}$. Then $N_{it} \cap \text{bd } S$ is a finite nonempty collection of 1-cells and $N_{it} \cap J_i$ is a finite nonempty collection of 1-cells all lying in the same component of bd N_{it};

(c) N_{it} meets R finitely;

(d) $N_{it} \subset i_S N_{i(t+1)}$ where i_S denotes point set interior relative to S;

(e) $N_{i0} = A_i$; and

(f) $S = \bigcup_{t=0}^{\infty} i_S N_{it}$.

Now let

$A = \{x \in R^2 | \frac{1}{2} \leq \|x\| \leq 1\}$,

$S_1 = \{x \in R^2 | \ \|x\| = 1\}$,

$S_0 = \{x \in R^2 | \ \|x\| = \frac{1}{2}\}$,

$\Sigma = \{x \in S_1 | x = e^{\pi i/2n}, n \text{ an integer}\} \cup \{(1, 0)\}$, and

$Y = \{(x, 0) | -1 \leq x \leq -\frac{1}{2}\}$.

Define V_i, $1 \leq i \leq 5$, by

$V_1 = A - (\Sigma \cup Y)$,

$V_2 = A - (S_1 \cup Y)$,

V_3 is a 2-cell,

$V_4 = A - \Sigma$, and

$V_5 = A - S_1$.

We now obtain D_c by making appropriate replacements. If S is of type (1), replace S by $e_{ij}(V_1)$ where e_{ij} is a homeomorphism of $A - Y$ onto $X_{ij} \cup L_{ij0}$ which carries S_0 onto X_{ij0} and Σ into $\text{int}(L_{ij0} \cap \text{bd } S)$. If S is of type (2), replace S by $e_{ij}(V_2)$ where e_{ij} is a homeomorphism of $A - Y$ onto $X_{ij} \cup L_{ij0}$ which carries S_0 onto X_{ij0}. If S is of type (3) use S. If S is of

type (4), replaces S by $e_i(V_4)$ where e_i is a homeomorphism of A onto $N_{i0} = A_i$ which carries S_0 onto bd E_i and Σ into $\operatorname{int}(N_{i0} \cap J_i) = \operatorname{int} \alpha_i$. Finally, replace each 2-cell H_n by $h_n(V_5)$ where h_n is an embedding of V_5 into H_n with $h_n(\operatorname{bd} V_5) = \operatorname{bd} H_n$. Then $D_c = K$ with replacements indicated above does generate L_c and a sketch of how this might be done is given below.

Let P be a 2-cell, $P \subset B_4$ such that $\operatorname{bd} P \cap \operatorname{bd} B_4 = \{x_1\}$. Let $\{K_i'\}_{i=1}^{\infty}$ be a sequence of 2-cells contained in $P \cap \operatorname{int} B_4$ such that $K_1' \cap \operatorname{bd} P$ is a 1-cell, $K_i' \subset i_P K_{i+1}'$ for all i and $\operatorname{int} B_4 = \bigcup_{i=1}^{\infty} \operatorname{int}((B_4 - P) \cup K_i')$. Note that $\operatorname{int} K_i \subset \operatorname{int} K_{i+1}$ where $K_i = (B_4 - P) \cup K_i'$. If the number of peripheral points t is finite and $1 \leq t' \leq t$ is the number of peripheral points contained in $\operatorname{int} B_3$, let $\{W_q'\}_{q=1}^{\infty}$ be defined by $W_q' = \bigcup_{p=1}^{t'} N_{pq}$. If $t = t'$ is infinite, let

$$W_q' = \left(\bigcup_{p=1}^{q} N_{pq}\right) \cup \left(\bigcup_{p=q+1}^{\infty} N_{p0}\right) = \left(\bigcup_{p=1}^{q} N_{pq}\right) \cup \left(\bigcup_{p=q+1}^{\infty} A_i\right).$$

Now set $PB_3 = \operatorname{int} B_3 - (\bigcup_{i=1}^{t'} \operatorname{int} E_i)$ and $W_q = PB_3 \cup W_q'$. Then for all $q \geq 1$, $W_q \subset i_{L_c} W_{q+1}$ and $\operatorname{int} B_3 \cap L_c = \bigcup_{q=1}^{\infty} i_{L_c} W_q$. Now define $\{G_k'\}_{k=1}^{\infty}$ by

$$G_k' = \Delta \cup \left(\bigcup_{j=1}^{4} (L_{ijk} \cup X_{ij})\right) \cup W_k \cup K_k$$

where $\Delta = S^2 - (\bigcup_{j=1}^{\alpha(M)} \operatorname{int} P_j)$ and where $1 \leq \alpha(M) \leq 3$ and i depend on M. Therefore for all k, $G_k' \subset i_{L_c} G_{k+1}'$ meets only a finite number of 2-cells from the sequence Γ_1 and Γ_2 and these are contained in $\operatorname{int} G_k'$. Furthermore, $L_c - (B_1 \cup B_2) = \bigcup_{k=1}^{\infty} i_{L_c} G_k'$. Now define

$$G_k = G_k' \cup (B_1 \cap L_c) \cup (B_2 \cap L_c).$$

Then $G_k \subset i_{L_c} G_{k+1}$ and $L_c = \bigcup_{k=1}^{\infty} i_{L_c} G_k$. We now observe that there exists a sequence of embeddings $\{\alpha_k\}_{k=1}^{\infty}$ of D_c into L_c such that for all $k \geq 1$, $\alpha_k(D_c)$ is open in L_c; $G_k \subset \alpha_k(D_c) \subset i_{L_c} G_{k+1}$; and α_k carries the sequences Γ_1' and Γ_2' onto the subsequences of Γ_1 and Γ_2 which are contained in G_{k+1} and hence in $i_{L_c} G_{k+1}$. Therefore D_c generates L_c and "respects Γ_1 and Γ_2". To obtain a domain D which generates M we need only remove the interiors of the 2-cells C_k' and D_k' in the sequences Γ_1' and Γ_2' and then appropriately identify the boundaries of the C_k'. Also it is clear from considering cases 1–4 that we can find a countably infinite collection of surfaces $\{D_k\}_{k=1}^{\infty}$ such that if M is a noncompact surface with nonempty

boundary, then some surface $D_{g(M)}$ generates M. Therefore, using this and the results of [3], it follows that the domain rank of any surface is at most countably infinite.

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