THE DOMAIN RANK OF A SURFACE IS COUNTABLE

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ABSTRACT. In this paper it is shown that any surface has at most countably infinite domain rank.

1. Introduction. For the convenience of the reader we first recall some of the definitions found in [1], [2], and [3]. A surface $M$ is a connected, separable, metrizable 2-manifold and a domain $D$ of a surface $M$ is an open connected subset of $M$. If $M$ is a surface, a set of surfaces $G^*$ is called a set of generating domains for $M$ if given any proper domain $D$ of the surface $M$, there is an element $G_D$ of $G^*$ such that $D = \bigcup_{k=1}^{\infty} D_k$ where for all $k$, $D_k$ is a domain of $M$, $D_k \subset D_{k+1}$, and $D_k$ is homeomorphic to $G_D$ (in such case we say that $G_D$ generates $D$ or that $D$ is an open monotone union of $G_D$). The domain rank of $M$, denoted by $DR(M)$, is defined by

$$DR(M) = \text{g.l.b.} \{|G^*| : G^* \text{ is a set of generating domains for } M\},$$

where $|G^*|$ denotes the cardinality of $G^*$.

The study of the domain rank of a surface is naturally motivated by the fact established in [1] that any domain $D$ of the plane $R^2$ is generated by $R^2 - \{(0, n) \mid n \text{ is a positive integer}\}$. This paper completes the study of this problem which has been previously considered in [1], [2], and [3].

2. Proof of the Theorem. Let $M$ be a noncompact surface, $bd M \neq \emptyset$. Let $s(M)$ be the number of components of $bd M$ which are 1-spheres. If $s(M) = 0$, let $M_c = M$; otherwise let $s(M) = s$, $1 \leq s \leq \infty$, and let $\{S_j\}_{j=1}^{s}$ be the collection of boundary components which are homeomorphic to $S^1$. Let $Q = \bigcup_{j=1}^{s} D_j$ be a disjoint union of 2-cells, $f: bd Q \to M$ be a continuous function which maps $bd D_j$ homeomorphically onto $S_j$, and $M_c$ be the surface defined by $M_c = Q \cup f$. We will identify $M_c$, $S_j$, and $D_j$ with their embedded images under the identification $p: Q + M_c$ and thus $M = M_c - (\bigcup_{j=1}^{s} \text{int } D_j)$. Let $N = bd M_c \times [0, 1) \cup g M_c$ where $g(x, 0) = x$ for...
all $x \in bd M_c$. Again, identify $bd M_c \times [0, 1)$ and all subsets of $M_c^-$ with their embedded images under the identification $q : bd M_c \times [0, 1) + M_c \rightarrow N$. As in [3], let $(X, Y, Z)$ denote the ideal boundary of the open surface $N$. Thus in view of Theorem 1.1 of [3], we may assume that $N$ is obtained from $S^2$ by removing $X$ and then removing the interiors of a finite or infinite sequence $\{C^+_k\}$ of pairwise disjoint 2-cells and properly identifying the boundaries of these 2-cells. Furthermore, it is clear that we may assume that each of these 2-cells does not meet $(bd M_c \times [0, 1) \cup (\bigcup_{j=1}^s D_j))$.

Let $W = bd M_c \times (0, 1)$ and $L_c = S^2 - (X \cup W)$. Then we can obtain $M_c$ from $L_c$ by removing the interiors of the 2-cells $C^+_k$ and properly identifying the boundaries.

Now consider a point $b \in$ ideal boundary $X$ of $N$. We will say $b$ is peripheral to $M$ $\iff$ there exists some component $K$ of $bd M_c = bd L_c$ such that the closure of $K$ in $S^2$ is $K \cup \{b\}$. If $b$ is peripheral to $M$, then a boundary leaf of $M$ at $b$ is any closed set of the form $K \cup \{b\}$ where $K$ is a boundary component of $M_c$.

We now wish to construct a domain $D$ which generates $M$. Let $p$ denote the genus of $M$, $0 \leq p \leq \infty$; $q$ the class of $N =$ class of int $M$, $1 \leq q \leq 4$; $s = s(M)$ the number of components of bd $M$ which are 1-spheres, $0 \leq s \leq \infty$; and $t$ the number of points $b \in$ ideal boundary of $N$ which are peripheral to $M$, $0 \leq t \leq \infty$. Let $F$ be the closed subset of $S^2$ defined by

$$F = X \cup bd M_c \times [0, 1) = X \cup W \cup bd L_c.$$ 

Assume that the sequence of disks $\Gamma_1 = \{C^+_{k}\}^{r}_{k=1}$, $0 \leq r \leq \infty$, is chosen as in the remark following Theorem 1.1 of [3]. If $p$ is finite, then $r$ is finite and we set $\Gamma_1' = \Gamma_1$. If $p$ is infinite, then as in the proof of Theorem 2.1 of [3] we may choose an appropriate subsequence $\Gamma_1' = \{C^+_{k}\}^{\infty}_{k=1}$ of $\Gamma_1$ which converges to a point $x_1 \in X$. Furthermore, we may assume that this subsequence is chosen such that for all cases, $0 \leq p \leq \infty$, we can construct a 2-cell $B_1$ in $S^2$ such that $B_1 \cap F = x_1$, $B_1 \cap (\bigcup_{j=1}^s D_j) = \emptyset$, $C^+_k \subset$ int $B_1$ for all $k$, and $C^+_k \cap B_1 \neq \emptyset$ $\iff$ $C^+_k \in \Gamma_1'$.

Now let $\Gamma_2 = \{D^+_j\}^{s}_{j=1}$ be the collection of 2-cells corresponding to the 1-sphere boundary components of $M$. If $s$ is finite, let $\Gamma_2' = \Gamma_2$. If $s = \infty$, then we can find a subsequence $\Gamma_2' = \{D^+_j\}^{\infty}_{j=1}$ which converges to some unique point $x_2 \in X$. Furthermore, we may assume this subsequence is chosen so that for all cases, $0 \leq s \leq \infty$, we can construct a 2-cell $B_2$ in $S_2$ such that $B_2 \cap F = x_2 \in X$, $B_2 \cap (\bigcup_{k=1}^t C^+_k) = \emptyset$, $D^+_j \subset$ int $B_2$ for all $j$. 

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$D_j \cap B_2 \neq \emptyset \Leftrightarrow D_j \in \Gamma_j'$, and $B_1 \cap B_2 \neq \emptyset \Rightarrow x_1 = x_2$ and $B_1 \cap B_2 = \{x_1\} = \{x_2\}$.

We now consider the points $b \in X$ which are peripheral to $M$. Observe that for each component $K_j$ of $\text{bd} \ M_c = \text{bd} \ L_c$, the closure of $[0, 1) \times K_j = [0, 1) \times K_j \cup \{b_j\}$ where $b_j \in X$. If each set $F_j = \text{closure} \ of \ [0, 1) \times K_j$ in $S_2$ is shrunk to the point $b_j$, we obtain $S_2$ and $X$ represents the ideal boundary of $\text{int} \ M_c$. Let

$$T = [0, 1) \times \text{bd} \ M_c \cup \left( \bigcup_{j=1}^{s} D_j \right) \cup \left( \bigcup_{k=1}^{r} C_k \right).$$

If the number $t$ of peripheral points is infinite, let $\Omega = \{b_i\}_{i=1}^{\infty}$ be a sequence of peripheral points such that $b_i \neq x_1$ or $x_2$ for all $i$ and $\Omega$ has a unique limit point $x_3 \in X$. Then we can construct a sequence $\Gamma_3 = \{E_i\}_{i=1}^{\infty}$ of pairwise disjoint 2-cells with the following properties:

(a) $b_i \in \text{int} \ E_i$;
(b) $E_i \cap B_v = \emptyset$ for all $i$ and $v = 1$ and $2$;
(c) there is a closed 2-annulus $A_i \subset E_i$ with $\text{bd} \ E_i \subset A_i$ such that $A_i \cap T = \alpha_i$, $\alpha_i$ a closed subset of some boundary leaf at $b_i$, $\alpha_i$ homeomorphic to $[0, 1]$ and $b_i \notin \alpha_i$; and
(d) $\text{cl} \left( \bigcup_{i=1}^{\infty} E_i \right) = \left( \bigcup_{i=1}^{\infty} E_i \right) \cup \{x_3\}$, $x_3 \in X$.

For each $i$, let $F_i = \text{cl} \left( E_i - A_i \right)$. We may further assume that $\Gamma_3$ is chosen so that we can construct a 2-cell $B_3$ of $S^2$ such that

(e) $E_i \subset \text{int} \ B_3$ for all $i$;
(f) $(B_3 - \left( \bigcup_{i=1}^{\infty} F_i \right)) \cap \{x_3\}$;
(g) $(B_3 - \left( \bigcup_{i=1}^{\infty} F_i \right)) \cap T = \emptyset$; and
(h) $B_3 \cap B_v \neq \emptyset$ for $v = 1$ or $2 \Rightarrow x_3 = x_v$ and $B_3 \cap B_v = \{x_v\}$.

If $t$ is finite, and $t' \geq 1$, where $t'$ is the cardinality of the set of peripheral points $\{x_1, x_2\}$, we can similarly construct 2-cells $\{E_i\}_{i=1}^{t'}$, with properties (a)–(c) and a 2-cell $B_3$ with properties analogous to (e)–(h) above where $x_3$ is some point of $X$. If $t' = 0$, construct $B_3$ satisfying appropriate properties similar to (f) and (h).

Finally, construct a 2-cell $B_4$ such that $B_4 \cap X = \{x_1\}$, $B_4 \cap T = \emptyset$, and $(B_4 - \{x_1\}) \cap B_v = \emptyset$, $v = 1, 2, 3$; and let $\{H_n\}_{n=1}^{\infty}$ be a disjoint sequence of 2-cells contained in $\text{int} \ B_4$ which converge to $x_1$.

We will say a 2-cell $B$ contained in a 2-cell $C$ is 1-cell proper in $C \Leftarrow B \cap \text{bd} \ C$ is a 1-cell. Let $Z = T \cup \left( \bigcup_{v=1}^{4} B_v \right)$. Then we can find a set $Q$ of disjoint 2-cells $\{Q_1, Q_2, Q_3\}$ with $Q_i = Q_j$ if $x_i = x_j$ such that
$x_i \in \text{int } Q_i$ for all $i$ and such that there is a 2-cell $P_i \subset \text{int } Q_i$ with $x_i \in \text{int } P_i$ and $(Q_i - \text{int } P_i) \cap Z = \emptyset$. Furthermore, we may assume the following depending on the relation between $x_1, x_2,$ and $x_3$:

Case 1. $x_1, x_2,$ and $x_3$ are all distinct. $B_v$ is 1-cell proper in $P_v$, $v = 1, 2, 3,$ and $B_4$ is 1-cell proper in $P_1$.

Case 2. $x_1 = x_2$. $B_3$ is 1-cell proper in $P_3$ and $B_v$ is 1-cell proper in $P_1$, $v = 1, 2,$ and 4.

Case 3. $x_2 = x_3$. $B_1$ and $B_4$ are 1-cell proper in $P_1$ and $B_2$ and $B_3$ are 1-cell proper in $P_2$.

Case 4. $x_1 = x_2 = x_3$. $B_v$ is 1-cell proper in $P_1$ for $1 \leq v \leq 4$.

By joining (if necessary) the 2-cells of the set $Q$ together by 2-cells lying in $S^2 - (T \cup X)$ we obtain a 2-cell $K$ which, depending on the case, has the following form:
THE DOMAIN RANK OF A SURFACE IS COUNTABLE 487

We now consider surfaces \( S \) of genus 0 contained in the surface \( L_c \) which take one of the following forms:

(i) \( T_{ij} \cap L_c, 1 \leq i, j \leq 4; \)
(ii) \( (S^2 - K) \cap L_c; \) or
(iii) \( E_i \cap L_c, 0 < i < M(i), \) where \( M(i) \) depends on \( M. \)

Note that for any of the above surfaces \( S, C_k \cap S \neq \emptyset \iff C_k \subset \text{int} S \) and \( D_j \cap S \neq \emptyset \iff D_j \subset \text{int} S \) and that the union of all \( C_k \) and \( D_j \) which meet a fixed \( S \) nonvoidly is closed in \( S. \) Also \( \text{bd} T_{ij}, \text{bd}(S^2 - K), \) and \( \text{bd} E_i \) do not meet \( \text{bd} L_c. \)

We now describe how to obtain a domain \( D_c \) which generates \( L_c \) and "respects the collection of 2-cells \( \Gamma_1 \) and \( \Gamma_2 \". A domain \( D \) which generates \( M \) is then easily obtained from \( D_c. \)

To obtain \( D_c, \) it is necessary to replace each surface \( S \) of the type mentioned by an appropriate surface of genus 0. Henceforth we will say that a surface \( Q \subset S^2 \) meets

\[
R = \left( \bigcup_{j=1}^{s} D_j \right) \cup \left( \bigcup_{k=1}^{r} C_k \right)
\]

finitely \( \iff \) at most a finite number of elements of \( \{D_j, C_k\} \) intersect \( Q \) nonvoidly and in such case are contained in \( \text{int} Q. \) Note that any surface \( S \) of the type mentioned above has one of the following properties:

(1) \( S = T_{ij} \cap L_c \) and \( T_{ij} \) contains a boundary leaf at \( x_v, 1 \leq v \leq 3, \) \( x_v \in \text{bd} T_{ij}. \) Let \( X_{ij} \) be a closed subset of \( S \) such that \( X_{ij} = R \times [0, 1], \) closure of \( X_{ij} \) in \( S^2 \) is \( X_{ij} \cup \{x_v\}, \) \( \text{bd} T_{ij} - \{x_v\} \) corresponds to \( R \times \{0\} = X_{ij0} \) and \( X_{ij} \cap T = \emptyset. \) Let \( X_{ij1} \) denote the boundary component of \( X_{ij} \) corresponding to \( R \times \{1\}. \) Then we can find a sequence of compact surfaces \( \{L_{ijt}\}_{t=0}^{\infty} \) of genus 0 with the following properties:

(a) \( L_{ijt} \cap X_{ij} = L_{ijt} \cap X_{ij1} \) is a 1-cell;
(b) \( L_{ijt} \cap \text{bd} S \) is a finite nonempty collection of 1-cells at least one of which lies in the same component of \( \text{bd} L_{ijt} \) as \( L_{ijt} \cap X_{ij1}; \)
(c) \( L_{ijt} \) meets \( R \) finitely;
(d) \( L_{ijt} \subset i_A L_{ij(t+1)} = \text{point set interior of} \ L_{ij(t+1)} \) relative to \( A = (S - X_{ij}) \cup X_{ij1}; \)
(e) \( L_0 \) is a 2-cell, \( L_0 \) does not meet \( R, \) and \( L_0 \cap \text{bd} S \) is a 1-cell; and

(f) \( S = X_{ij} \cup \left( \bigcup_{t=0}^{\infty} i_A L_{ijt} \right) = \bigcup_{t=0}^{\infty} i_S (X_{ij} \cup L_{ijt}). \)

(2) \( S = T_{ij} \cap L_c \) and \( T_{ij} \) contains no boundary leaf at \( x_v, 1 \leq v \leq 3, \)
Choose $X_{ij}$ as in (1). Then we can find a sequence of compact surfaces $\{L_{ijt}\}_{t=0}^\infty$ of genus 0 with properties (a), (c), (d), and (f) as in (1) but with (b) and (e) as follows:

(b) $L_{ijt} \cap \text{bd}\ S$ is a finite collection of 1-cells, none of which lie in the same component of $\text{bd}\ L_{ijt}$ as $L_{ijt} \cap X_{ij1}$; and

(e) $L_0$ is a 2-cell, $L_0$ does not meet $F$, and $L_0 \cap \text{bd}\ S = \emptyset$.

(3) $S = (S^2 - K) \cap L_c$ and $S$ is a 2-cell. Note that we may always assume that this is the case since if not, we can move any 2-cell contained in $\text{int}\ (S^2 - K)$ into $\text{int}\ (T_{ij1})$ moving only points inside an appropriate 2-cell and thus obtain the desired situation.

(4) $S = E_i \cap L_c$ and $E_i$ contains a boundary leaf at $b_i$, $b_i \in \text{int}\ F_i$. Then we can find a sequence of compact surfaces $\{N_{it}\}_{t=0}^\infty$ of genus 0 with the following properties:

(a) $N_{it} \cap \text{bd}\ E_i = \text{bd}\ E_i$ for all $t$;

(b) Let $J_i = \bigcup \{M|M$ is a leaf at $b_i\}$. Then $N_{it} \cap \text{bd}\ S$ is a finite nonempty collection of 1-cells and $N_{it} \cap J_i$ is a finite nonempty collection of 1-cells all lying in the same component of $\text{bd}\ N_{it}$;

(c) $N_{it}$ meets $R$ finitely;

(d) $N_{it} \subset i_S N_{i(t+1)}$ where $i_S$ denotes point set interior relative to $S$;

(e) $N_{i0} = A_i$; and

(f) $S = \bigcup_{t=0}^\infty i_S N_{it}$.

Now let

\[ A = \{x \in R^2|\frac{1}{2} \leq ||x|| \leq 1\}, \]

\[ S_1 = \{x \in R^2| ||x|| = 1\}, \]

\[ S_0 = \{x \in R^2| ||x|| = \frac{1}{2}\}, \]

\[ \Sigma = \{x \in S_1|x = e^{\pi i/2n}, n \text{ an integer}\} \cup \{(1,0)\}, \]

\[ Y = \{(x,0)|-1 \leq x \leq \frac{1}{2}\}. \]

Define $V_i$, $1 \leq i \leq 5$, by

\[ V_1 = A - (\Sigma \cup Y), \]

\[ V_2 = A - (S_1 \cup Y), \]

\[ V_3 \text{ is a 2-cell}, \]

\[ V_4 = A - \Sigma, \text{ and} \]

\[ V_5 = A - S_1. \]

We now obtain $D_c$ by making appropriate replacements. If $S$ is of type (1), replace $S$ by $e_{ij}(V_1)$ where $e_{ij}$ is a homeomorphism of $A - Y$ onto $X_{ij} \cup L_{ij0}$ which carries $S_0$ onto $X_{ij0}$ and $\Sigma$ into $\text{int}(L_{ij0} \cap \text{bd}\ S)$. If $S$ is of type (2), replace $S$ by $e_{ij}(V_2)$ where $e_{ij}$ is a homeomorphism of $A - Y$ onto $X_{ij} \cup L_{ij0}$ which carries $S_0$ onto $X_{ij0}$. If $S$ is of type (3) use $S$. If $S$ is of
THE DOMAIN RANK OF A SURFACE IS COUNTABLE 489

type (4), replaces $S$ by $e_i(V_4)$ where $e_i$ is a homeomorphism of $A$ onto $N_{i_0} = A_i$ which carries $S_0$ onto $\text{bd} E_i$ and $\Sigma$ into $\text{int}(N_{i_0} \cap J_i) = \text{int} \alpha_i$. Finally, replace each 2-cell $H_n$ by $h_n(V_5)$ where $h_n$ is an embedding of $V_5$ into $H_n$ with $h_n(\text{bd} V_5) = \text{bd} H_n$. Then $D_c = K$ with replacements indicated above does generate $L_c$ and a sketch of how this might be done is given below.

Let $P$ be a 2-cell, $P \subset B_4$ such that $\text{bd} P \cap \text{bd} B_4 = \{x_1\}$. Let $\{K_{i_j}\}_{i_j=1}^\infty$ be a sequence of 2-cells contained in $P \cap \text{int} B_4$ such that $K_{i_j} \cap \text{bd} P$ is a 1-cell, $K_{i_j} \subset i_pK_{i_j+1}$ for all $i$ and $\text{int} B_4 = \bigcup_{i=1}^\infty \text{int}((B_4 - P) \cup K_i)$. Note that $\text{int} K_{i_j} \subset \text{int} K_{i_j+1}$ where $K_{i_j} = (B_4 - P) \cup K_i$. If the number of peripheral points $t$ is finite and $1 \leq t' \leq t$ is the number of peripheral points contained in $\text{int} B_3$, let $\{W_{i_j}\}_{i_j=1}^\infty$ be defined by $W_{i_j} = \bigcup_{r=1}^t N_{i_j}$, where $i_{t_j} = i_{t_j+1}$. If $t = t'$ is infinite, set $W_{i_j} = \bigcup_{r=1}^t N_{i_j}$. Now define $\{G_{i_j}\}_{i_j=1}^\infty$ by $G_{i_j} = \Delta \cup \left( \bigcup_{r=1}^4 (L_{i_j} \cup X_{i_j}) \right) \cup W_{i_j} \cup K_{i_j}$

where $\Delta = S^2 - (\bigcup_{i=1}^{\alpha(M)} \text{int} P_i)$ and where $1 \leq \alpha(M) \leq 3$ and $i$ depend on $M$.

Therefore for all $k$, $G_k \subset i_{L_c}G_{k+1}$ meets only a finite number of 2-cells from the sequence $\Gamma_1$ and $\Gamma_2$ and these are contained in $\text{int} G_{i_j'}$. Furthermore, $L_c - (B_1 \cup B_2) = \bigcup_{k=1}^\infty i_{L_c}G_{k}$. Now define $G_k = G_k \cup (B_1 \cap L_c) \cup (B_2 \cap L_c)$. Then $G_k \subset i_{L_c}G_{k+1}$ and $L_c = \bigcup_{k=1}^\infty i_{L_c}G_k$. We now observe that there exists a sequence of embeddings $\{\alpha_k\}_{k=1}^\infty$ of $D_c$ into $L_c$ such that for all $k \geq 1$, $\alpha_k(D_c)$ is open in $L_c$; $G_k \subset \alpha_k(D_c) \subset i_{L_c}G_{k+1}$; and $\alpha_k$ carries the sequences $\Gamma_1$ and $\Gamma_2$ onto the subsequences of $\Gamma_1$ and $\Gamma_2$ which are contained in $G_{k+1}$ and hence in $i_{L_c}G_{k+1}$. Therefore $D_c$ generates $L_c$ and "respects $\Gamma_1$ and $\Gamma_2". To obtain a domain $D$ which generates $M$ we need only remove the interiors of the 2-cells $C_{i_j}'$ and $D_{i_j}'$ in the sequences $\Gamma_1'$ and $\Gamma_2'$ and then appropriately identify the boundaries of the $C_{i_j}'$. Also it is clear from considering cases 1–4 that we can find a countably infinite collection of surfaces $\{D_c\}_{c=1}^\infty$ such that if $M$ is a noncompact surface with nonempty
boundary, then some surface \( D_{g(M)} \) generates \( M \). Therefore, using this and the results of [3], it follows that the domain rank of any surface is at most countably infinite.

REFERENCES

