A NOTE ON THE RELATIONSHIP BETWEEN
WEIL AND CARTIER DIVISORS

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ABSTRACT. Using a generalized equivalence relation, a subquotient of the group of Weil divisors is shown to be isomorphic to the group of Cartier divisors modulo linear equivalence for a reduced subscheme of a projective space over a field. A difficulty of the nonreduced case is discussed.

Let $X$, $\mathcal{O}$ be a subscheme of a projective space over a field. A generalized equivalence relation is defined on the group of Weil divisors, and if $X$, $\mathcal{O}$ is reduced, a corresponding subquotient is shown to be isomorphic to the group of Cartier divisors modulo linear equivalence. The generalized equivalence for curves appears in [5]. Projectivity may be replaced by the conditions that any finite number of points of $X$ lie in an affine open of $X$ and that the nonregular locus of $X$, $\mathcal{O}$ is closed. An example is given which shows one difficulty of the nonreduced case.

By a prime ideal $p$ of $\mathcal{O}$ is meant a subsheaf $p$ of ideals of $\mathcal{O}$ such that for every open $U \subset X$, $\Gamma(U, p)$ is a prime ideal of $\Gamma(U, \mathcal{O})$. These are the points of $X$ by the usual correspondence.

For reducible schemes, the notation for zero-divisors developed in [2] is used. A zero prime ideal $N$ of $\mathcal{O}$ is a proper prime ideal of $\mathcal{O}$ such that for each open $U \subset X$, $\Gamma(U, N)$ is either $\Gamma(U, \mathcal{O})$ or consists entirely of zero divisors of $\Gamma(U, \mathcal{O})$. A divisorial prime ideal $p$ of $\mathcal{O}$ is a prime ideal $p$ of $\mathcal{O}$ which contains a zero prime ideal $N$ of $\mathcal{O}$ such that $p/N$ is of height one in $\mathcal{O}/N$.

Let $\mathcal{Q}/\mathcal{O}$ denote the total quotient sheaf of $\mathcal{O}$, and let $K = \Gamma(X, \mathcal{Q}/\mathcal{O})$. Let $\mathcal{D} = \Gamma(X, (\mathcal{Q}/\mathcal{O})^*)/\mathcal{O}^*$, the group of Cartier divisors, let $\mathcal{P} = \Gamma(X, (\mathcal{Q}/\mathcal{O})^*)/\Gamma(X, \mathcal{O}^*)$, and let $\mathcal{C} = \mathcal{D}/\mathcal{P}$. ($\mathcal{P}$ is the set of principal Cartier divisors, and $\mathcal{P}$ defines linear equivalence on $\mathcal{D}$.) Let $(f)$ denote the principal Cartier divisor defined by $f \in K^*$.

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Let $\mathcal{D} \subset X$ be the set of nondivisorial prime ideals $p$ of $\mathcal{O}$ such that if $f$ is a regular element of $\mathcal{O}_p$ then $\mathcal{O}_p f$ is an embedded prime ideal of $\mathcal{O}_p$. Let $\mathcal{C}$ be the union of $\mathcal{D}$ and the set of nonnormal (nonregular) divisorial prime ideals. If $X, \mathcal{O}$ is reduced, then $\mathcal{D}$ is the set of nondivisorial prime ideals of depth (grade) one, and both $\mathcal{D}$ and $\mathcal{C}$ are finite.

Let $D$ be the group of Weil divisors of $X$, the free abelian group generated by the divisorial prime ideals of $\mathcal{O}$. Let $p$ be a divisorial prime ideal of $\mathcal{O}$, let $f \in K^*$, let $f_1$ and $f_2$ be regular elements of $\mathcal{O}_p$ such that $f = f_1/f_2$, and define

$$\lambda_p(f) = \sum_{N \subset p} \left( \ell_{\mathcal{O}_p} \left( \frac{\mathcal{O}_p}{f_1 + N_p} \right) - \ell_{\mathcal{O}_p} \left( \frac{\mathcal{O}_p}{f_2 + N_p} \right) \right),$$

where the sum is over all zero prime ideals $N \subset p$ with $p/N$ of height one in $\mathcal{O}/N$. Let the principal Weil divisor defined by $f \in K^*$ be $\langle f \rangle = \Sigma_p \lambda_p(f) p$, where the sum is over all divisorial prime ideals $p$. Let $S$ be the set of all divisorial prime ideals which are contained in some element of $\mathcal{C}$. Let $D \setminus S$ be the free abelian group generated by those divisorial prime ideals not belonging to $S$. Let $I$ be the subgroup of $D$ of principal divisors $\langle f \rangle$ where $f \in K^*$ is such that $f_q \in \mathcal{O}_q^*$ for all $q \in \mathcal{C}$, consider $I$ as a subgroup of $D \setminus S$, and let $C = (D \setminus S)/I$. $(D/I)$ is the generalized class group of $X, \mathcal{O}$ and $I$ defines a generalized equivalence relation. Letting $P$ be the subgroup of $D$ of all principal divisors $\langle f \rangle$, $D/P$ is the class group of $X, \mathcal{O}$, and $P$ defines linear equivalence on $D$.

To construct the injection from $\mathcal{C}$ to $C$ for $X, \mathcal{O}$ reduced, only the following approximation lemma for Cartier divisors is needed. It is a variant of the usual approximation [3, Lemma 3, p. 166].

Recall that if $p_1, \ldots, p_s$ and $p$ are prime ideals of $\mathcal{O}$, then $p \subset p_1 \cup \cdots \cup p_s$ if and only if $p \subset p_i$ for some $i = 1, \ldots, s$. Let $q_1, \ldots, q_l$ also be prime ideals of $\mathcal{O}$. The condition below that

$$p \subset (p_1 \cup \cdots \cup p_s) \cap (q_1 \cup \cdots \cup q_l)$$

is equivalent to $p \subset p_i \cap q_j$ for some $i$ and $j$.

**Lemma.** Let $X, \mathcal{O}$ be a subscheme of a projective space over a field. Let $p_1, \ldots, p_s$ be prime ideals of $\mathcal{O}$, let $D$ be a Cartier divisor on $X$, and let $q_1, \ldots, q_l$ be prime ideals of $\mathcal{O}$ such that $D_p = (1)_p$ for every prime ideal $p$ of $\mathcal{O}$ with $p \subset (p_1 \cup \cdots \cup p_s) \cap (q_1 \cup \cdots \cup q_l)$. Then there is an
element $f$ of $K^*$ such that $(f)_{p_i} = D_{p_i}$ for $i = 1, \ldots, s$ and $(f)_{q_i} = (1)_{q_i}$ for $i = 1, \ldots, t$.

Proof. First assume the Lemma is true for $s = 1$, and induct on $s$. Assuming true for $s - 1$, there is an $f'' \in K^*$ with $(f'')_{q_i} = (1)_{q_i}$ for $i = 1, \ldots, t$ and $(f'')_{p_i} = D_{p_i}$ for $i = 1, \ldots, s - 1$. Then $(D - (f''))_{p_i} = (1)_{p_i}$ for $i = 1, \ldots, s - 1$, and $(D - (f''))_{q_i} = D_{q_i}$ for $i = 1, \ldots, t$. It follows that if $p$ is a prime ideal with

$$p \subset (p_1 \cup \cdots \cup p_{s-1} \cup q_1 \cup \cdots \cup q_t) \cap p_s$$

then $(D - (f''))_p = (1)_p$. Thus there is an $f' \in K^*$ such that $(f')_{p_i} = (D - (f''))_{p_i}$, $(f')_{q_i} = (1)_{q_i}$ for $i = 1, \ldots, s - 1$, and $(f')_{q_i} = (1)_{q_i}$ for $i = 1, \ldots, t$. Let $f = f''/f' \in K^*$.

Now let $s = 1$ and $p = p_1$. Let $A = \Gamma(U, \mathcal{O})$ where $U$ is an affine open subset of $X$ meeting each irreducible component of $X$ and containing the points $q_1, \ldots, q_t$ and $p$ which are to be considered as prime ideals of $A$. Let $g_1$ and $g_2$ be regular elements of $A$ such that $D_p = (g_1/g_2)_p$.

Let $P_1, \ldots, P_r$ be the associated prime ideals of $Ag_2$ for which $g_1/g_2 \notin A_{P_i}$. $P_i$ is not a zero prime ideal because $g_2 \notin P_i$. If $P_i \subset p$ then $D_{P_i} = (g_1/g_2)_{P_i} = (1)_{P_i}$ and $P_i \notin q_1 \cup \cdots \cup q_t$, and let $e_i \in P_i$ be a regular element of $A$ such that $e_i \notin q_1 \cup \cdots \cup q_t$. Or, if $P_i \notin p$, let $e_i \in P_i$ be a regular element of $A$ with $e_i \notin p$. There is an integer $n \geq 1$ such that $(e_1 \cdots e_r)^n g_1 \in Ag_2$. Let $h_1 = (e_1 \cdots e_r)^n g_1 g_2^{-1} \in A$, and let $h_2 = (e_1 \cdots e_r)^n$. Then $D_p = (h_1/h_2)_p$ and no isolated prime ideal of $Ah_1$ or $Ah_2$ is contained in $p \cap (q_1 \cup \cdots \cup q_t)$.

Let $Q_1, \cdots, Q_s$ be the associated prime ideals of $A0$ in $A$. For $j = 1, 2$, let $k_j$ be an element of $p$ contained in all the isolated prime ideals of $A_{P_i} h_j$ such that for $i = 1, \ldots, t$, $k_j \in q_i$ if and only if $h_j \notin q_i$, and such that for $i = 1, \ldots, r$, $k_j \in Q_i$ if $Q_i \notin q_1 \cup \cdots \cup q_t$. There is an integer $n \geq 1$ such that $k_j^n \in A_{P_i} h_j$ for $j = 1, 2$, for $k_j$ is contained in each associated prime ideal of $A_{P_i} h_j$. For $j = 1, 2$, let $f_j = h_j + k_j^{n+1}$. Then $f_j \notin q_1 \cup \cdots \cup q_t, f_j$ is a regular element of $A, 1 + h_j^{-1} k_j^{n+1}$ is a unit of $A_p$, and $(f_1/f_2)_p = D_p$.

Let $f = f_1/f_2$. Q.E.D.

The injection $\phi: \mathbb{C} \to C$ is to be constructed. Let $\alpha \in \mathbb{C}$. By the Lemma there is a divisor $D$ in $\alpha$ such that $D_p = (1)_p$ for all $p \in \mathbb{C}$, and therefore $D_p =
(1) for all $p \in S$. For a divisorial prime ideal $p$, let $f_1$ and $f_2$ be regular elements of $\mathcal{O}_p$ with $D_p = \langle f_1/f_2 \rangle_p$, and define

$$
\lambda_p(D) = \sum_{N \in \mathcal{O}_p} \left( \ell_{\mathcal{O}_p} \left( \frac{\mathcal{O}_p}{\mathcal{O}_p f_1 + N} \right) - \ell_{\mathcal{O}_p} \left( \frac{\mathcal{O}_p}{\mathcal{O}_p f_2 + N} \right) \right),
$$

where the sum is over all zero prime ideals $N$ of $\mathcal{O}_p$ such that $\mathcal{O}_p p/N$ is of height one in $\mathcal{O}_p/N$. Let

$$
\lambda D = \sum_{p \notin S} \lambda_p(D) p \in D \setminus S,
$$

where the sum is over all divisorial prime ideals $p$ not contained in $S$. If $f \in K^*$ and $(f)_p = (1)_p$ for all $p \in \mathfrak{C}$, then $\lambda(f) = \langle f \rangle \in I$. Letting $\phi \alpha$ be the image of $\lambda D$ in $C = (D \setminus S)/I$, $\phi : \mathfrak{C} \to C$ is a well-defined homomorphism.

**Theorem.** Let $X$, $\mathcal{O}$ be a reduced subscheme of a projective space over a field. The length homomorphism $\phi : \mathfrak{C} \to C$ is injective. The image of $\phi$ is the subgroup of locally principal elements of $C$, which is a subquotient of $D$. Furthermore $\phi : \mathfrak{C} \to C$ is surjective if and only if $\mathcal{O}_p$ is factorial for all prime ideals $p$ of $\mathcal{O}$ which are contained in no element of $\mathfrak{C}$.

**Proof.** The usual argument follows ([4, Propositions, pp. 65, 66]). Let $\mathfrak{D} \setminus \mathfrak{C}$ be the subgroup of $\mathfrak{D}$ of divisors $D$ such that $D_p = (1)_p$ for all $p \in \mathfrak{C}$. Let $D \in \mathfrak{D} \setminus \mathfrak{C}$ with $\lambda D = \langle 1 \rangle$. Let $p$ be a prime ideal of $\mathcal{O}$, and let $f_1$ and $f_2$ be regular elements of $\mathcal{O}_p$ such that $D_p = \langle f_1/f_2 \rangle_p$. If $q \subset p$ is a prime ideal contained in $\mathcal{C}$, then $D_q = (1)_q$ and $\mathcal{O}_q f_1 = \mathcal{O}_q f_2$. If $q \subset p$ is a divisorial prime ideal not contained in $\mathcal{C}$, then $\ell_{\mathcal{O}_q} (\mathcal{O}_q f_1 / \mathcal{O}_q f_2) = \ell_{\mathcal{O}_q} (\mathcal{O}_q / \mathcal{O}_q f_1)$ and, because $\mathcal{O}_q$ is normal of Krull dimension one, $\mathcal{O}_q f_1 = \mathcal{O}_q f_2$. Thus $\mathcal{O}_q f_1 = \mathcal{O}_q f_2$ for all depth one (grade one) prime ideals $q$ of $\mathcal{O}_p$, and

$$
\mathcal{O}_p f_1 = \bigcap_q \mathcal{O}_q f_1 = \bigcap_q \mathcal{O}_q f_2 = \mathcal{O}_p f_2.
$$

Hence $\mathcal{O}_p f_1 = \mathcal{O}_p f_2$ for all prime ideals $p$ of $\mathcal{O}$, $f_1/f_2 \in \mathcal{O}_p^*$, $D = \langle 1 \rangle$, and $\lambda : \mathfrak{D} \setminus \mathfrak{C} \to D \setminus S$ is injective.

Now, to show that $\phi : \mathfrak{C} \to C$ is injective, let $\alpha \in \mathfrak{C}$ with $\phi \alpha = 0 \in C$, and let $D \in \alpha$ be such that $D_q = (1)_q$ for all $q$ in $\mathcal{C}$. Let $f \in K^*$ be such that $f \in \mathcal{O}_q^*$ for all $q$ in $\mathcal{C}$ and $\lambda D = \langle f \rangle$. Then by the injectivity of $\lambda$ above, $D = \langle f \rangle$, and $\alpha = 0$.

$\phi : \mathfrak{C} \to C$ is surjective if and only if $\lambda : \mathfrak{D} \setminus \mathcal{C} \to D \setminus S$ is surjective.

This is true if and only if the group of Cartier divisors is equal to the group
of Weil divisors for each local ring $\mathcal{O}_p$ where $p$ is a prime ideal of $\mathcal{O}$ which is contained in no element of $\mathcal{C}$, which is in turn equivalent to each $\mathcal{O}_p$ being factorial for all these prime ideals $p$. Q.E.D.

If $\mathcal{O}$ is not reduced, $\mathcal{C}$ may no longer be finite, and the construction used to define $\phi$ may not be applicable. For example, let $R = k[x, y, z]/(x^2, xy)$ where $k$ is a field. The reduction of $R$, $R/Rx = k[y, z]$, is the polynomial ring in two variables over $k$. Let $p$ be a prime element of $k[z]$. $Rp$ has as associated prime ideals $(p, x)$ and $(p, x, y)$, for $(x, y)$ is an embedded component of $(0)$ in $(k[z]/(p))[x, y]/(x^2, xy) \cong R/Rp$.

Thus $(p, x, y) \in \mathcal{D}$, and $\mathcal{D}$ is infinite. The similar projective example given by the homogeneous ring $k[W, X, Y, Z]/(X^2, XY)$ is such that every homogeneous height one prime ideal is contained in an element of $\mathcal{D}$.

REFERENCES


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