AN EXTENSION OF THE ERDÖS-RÉNYI
NEW LAW OF LARGE NUMBERS

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ABSTRACT. If \( S_n \) is the \( n \)th partial sum of a sequence of independent, identically distributed random variables \( X_1, X_2, \ldots \) such that \( E(X_1) = 0 \) and \( E(\exp(tX_1)) < \infty \) for some nonempty interval of \( t \)'s, then, for a wide range of positive numbers \( \lambda \), Erdös and Rényi (1970) showed that \( \Sigma(N, [C(\lambda)\log N]) \) converges with probability one to \( \lambda \) as \( N \to \infty \), where \( \Sigma(N, K) \) is the maximum of the \( n - K + 1 \) averages of the form \( K^{-1}(S_{n+K} - S_n) \) for \( 0 \leq n \leq N - K \), and \( C(\lambda) \) is a known constant depending on \( \lambda \) and the distribution of \( X_1 \). The objective of the present article is to state and prove the Erdös-Rényi theorem for the \( n - K + 1 \) "averages" of the form \( K^{-1/r}(S_{n+K} - S_n) \), where \( 1 < r < 2 \). This form of the Erdös-Rényi theorem arises from the extended form of the strong law of large numbers which asserts that, if \( E(|X_1|^r) < \infty \) for some \( r \), \( 1 \leq r < 2 \), and \( E(X_1) = 0 \), then \( n^{-1/r}S_n \) converges with probability one to \( 0 \) as \( n \to \infty \).

0. Introduction. If \( \{X_n : 1 \leq n < \infty \} \) is a sequence of independent, identically distributed (i.i.d.) random variables, with partial sums \( S_n = \sum_{k=1}^{n} X_k \), then there are \( N - K + 1 \) successive averages of the form \( K^{-1}(S_{n+K} - S_n) \), one for each value of \( n \) between \( 0 \) and \( N - K \), inclusive. We denote the largest of these averages by \( \Sigma(N, K) \), i.e.

\[
\Sigma(N, K) = \max \{K^{-1}(S_{n+K} - S_n) : 0 \leq n \leq N - K\}.
\]

Then, by the Borel-Cantelli lemma, \( \operatorname{P}\lim_{N \to \infty} \Sigma(N, 1) = \mu \} = 1 \) if \( \mu \leq \infty \) is the essential supremum of \( X_1 \). If \( E(X_1) = 0 \), the strong law of large numbers asserts that \( \operatorname{P}\lim_{N \to \infty} \Sigma(N, N) = 0 \} = 1 \). What happens to \( \Sigma(N, K) \) for values of \( K \) between \( 1 \) and \( N \) was the subject of the 1970 article [4] by Erdös and Rényi. If \( X_1 \) has moment-generating function \( \phi(t) = E(\exp(tX_1)) < \infty \) for a sufficiently large interval of \( t \)'s, then for every \( \lambda \) between \( 0 \) and \( \mu \) (exclusive

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of the endpoints), Erdös and Rényi showed that there exists a constant $C(\lambda)$, whose value can be determined, depending on $\lambda$ and the distribution of $X_1$, such that

$$P \left\{ \lim_{N \to \infty} \Sigma(N, [C(\lambda) \log N]) = \lambda \right\} = 1.$$ 

Here $[y]$ denotes the greatest integer $\leq y$. Erdös and Rényi called their result "a new law of large numbers" because of the fundamental relation it bears to the strong law of large numbers. The theorem in [2] extends the Erdös-Rényi result to the case of weighted sums of i.i.d. random variables.

It is the objective of the present article to extend the Erdös-Rényi theorem to averages of the form $K^{-1/r}(S_{n+K} - S_n)$ for $1 < r < 2$. Toward this end, we define

$$\Sigma_r(N, K) = \max\{K^{-1/r}(S_{n+K} - S_n): 0 < n < N - K\}.$$ 

Again $P\lim_{N \to \infty} \Sigma_r(N, 1) = \mu = 1$ and, if $E(X_1) = 0$ and $E(|X_1|^r) < \infty$, the strong law of large numbers asserts that $P\lim_{N \to \infty} \Sigma_r(N, N) = 0 = 1$. The strong law of large numbers in the form $P(n^{-1/r}S_n \to 0) = 1$ for $1 < r < 2$ can be found in Loève’s text [7, p. 243]. In the present article, we intend to show that, for every positive $\lambda$, $0 < \lambda < \infty$.

$$P \left\{ \lim_{N \to \infty} \Sigma_r(N, [(2\lambda^{-2} \log N)^{r/(2-r)}]) = \lambda \right\} = 1$$

as long as there exists some nondegenerate interval of $t$'s, containing the origin in its interior, for which $\phi(t) = E(\exp(tX_1)) < \infty$. Note that, in the present situation, $\phi(t)$ is required to exist only in a nondegenerate interval, not necessarily a sufficiently large one, and that the constant $2\lambda^{-2}$ depends only on $\lambda$ and is independent of the distribution of $X_1$.

Erdös and Rényi based their proof on the large deviation theorem of Cramér [3], in the form due to Bahadur and Ranga Rao [1], which asserts that

$$P(S_n > \lambda n) \sim (2\pi n)^{-1/2} \rho^n b_n,$$

where $\rho = \exp(-1/C(\lambda))$, $\{b_n: 1 \leq n < \infty\}$ is a bounded sequence, and the symbol "$\sim$" indicates that the ratio of the two sides tends to 1 as $n \to \infty$. To provide a point of departure for the theorem of the present article, we need a large deviation theorem also, one due to Petrov [8], which can be found in the most easily usable form in the monograph of Ibragimov and Linnik [6]. Petrov’s theorem, which is a generalization of and has the same form as the original Cramér theorem, implies in our situation, for $1 < r < 2$, that

$$P(S_n > \lambda n^{1/r}) \sim (2\pi n)^{-1/2} \rho^n b_n \exp\{-1/2\lambda^{-2}\exp[-1/2\lambda^{-2}a(1 + o(1))\}]$$
as \( n \to \infty \), where \( \alpha = (2 - r)/r \). Note that if \( r = 1 \), then \( \alpha = 1 \), and the above statement is somewhat reminiscent of the Bahadur-Ranga Rao result.

Petrov's large deviation theorem and the corollary of it that we use are recorded in §1. §2 contains the extension of the Erdős-Rényi theorem. In §3, we briefly compare the main theorem of this article with the original Erdős-Rényi theorem, focusing on the relationship between \( C(\lambda) \) and \( 2\lambda^{-2} \).

1. Petrov's theorem and its corollary. We consider throughout the paper a sequence of nondegenerate i.i.d. random variables \( \{X_n : 1 \leq n < \infty \} \) such that \( E(X_1) = 0 \), \( \Var(X_1) = 1 \), and \( \phi(t) = E(\exp(tX_1)) < \infty \) for \( |t| < B \), where \( 0 < B \leq \infty \).

For the theorem of the present article, we need no condition on the moment-generating function (m.g.f.) \( \phi \) beyond its existence in a nondegenerate interval containing the origin in its interior. Bahadur and Ranga Rao, and therefore Erdős and Rényi, had to require that the function \( Q(t) = \phi'(t)/\phi(t) \) take on the value \( \lambda \) at some point \( t = t_\lambda \). (An analogous condition is required in the case of weighted sums that is studied in [2].) That assumption is vital in their situation, however, because \( C(\lambda) \) turns out to be the reciprocal of \( \lambda t_\lambda - \log \phi(t_\lambda) \). Moreover, a sufficient condition for the existence of \( t_\lambda = Q^{-1}(\lambda) \) is that \( \phi(t) < \infty \) for all real \( t \) and \( P(X_1 > \lambda) > 0 \). Now, if \( P(X_1 > \lambda) = 0 \), particularly if \( \lambda > \mu \), the essential supremum of \( X_1 \), then \( \Sigma(N, K) \leq \mu < \lambda \) for all \( N \) and \( K \) so that \( \lim_{N \to \infty} \Sigma(N, K) = \lambda \) is not possible. In our case, on the other hand, \( \Sigma_r(N, K) > \lambda \) is possible even if \( \lambda > \mu \), so no such condition relating \( \phi \) and \( \lambda \) is necessary.

We consider a sequence of numbers \( \{z_n : 1 \leq n < \infty\} \) such that \( z_n \to \infty \) and \( n^{-\frac{1}{2}}z_n \to 0 \) as \( n \to \infty \), and we want to study the probability \( P(n^{-\frac{1}{2}}S_n \geq z_n) \), where \( S_n = \sum_{k=1}^{n} X_k \). We will be especially interested in the case \( z_n = \lambda n^{\alpha/2} \), where \( \alpha = (2 - r)/r \) for \( 1 < r < 2 \). In the case \( r = 2 \), \( \alpha = 2 \) and we know by the most elementary form of the central limit theorem that

\[
P(n^{-\frac{1}{2}}S_n \geq \lambda) \to 1 - \Phi(\lambda) = (2\pi)^{-\frac{1}{2}} \int_{\lambda}^{\infty} \exp(-u^2/2)du \quad \text{as } n \to \infty.
\]

For \( r < 2 \), \( \alpha > 0 \) so that \( P(n^{-\frac{1}{2}}S_n \geq \lambda n^{\alpha/2}) \) tends to 0 as \( n \to \infty \). The rate at which this probability tends to 0 when \( r = 1 \) is the subject of Cramér [3] and Bahadur and Ranga Rao [1]. We now deal with the case \( 1 < r < 2 \).

Under the conditions discussed above, we have the following large deviation theorem, which can be found in [6, p. 171]:

Theorem 1.1 (Petrov). If \( \phi(t) < \infty \) for \( |t| < B \), where \( 0 < B \leq \infty \), and

\[
z_n \to \infty \text{ and } n^{-\frac{1}{2}}z_n \to 0 \quad \text{as } n \to \infty,
\]

then
P(n^{-1/2}S_n \geq z) = \{1 - \Phi(z_n)\} \exp \{n^{-1/2}z^3 \lambda(n^{-1/2}z_n)\} \{1 + O(n^{-1/2}z_n)\},

where

\Phi(z) = \left(2\pi\right)^{-1/2} \int_{-\infty}^{z} \exp(-t^2/2) \, dt

and \lambda(z) is the Cramér series.

In Petrov's theorem, the Cramér series \lambda(z) = \sum_{k=0}^{\infty} \lambda_k z^k is a power series whose coefficients depend on the moments and semi-invariants of \(X_1\) and which converges for all sufficiently small values of \(z\), the radius of convergence depending on \(\phi\). Further details about the series are available, for example, in Cramér's original article [3] and in Chapter 7 of Ibragimov and Linnik's book [6]. In addition, by a fact proved in Feller's text [5, p. 175], \(z_n \to \infty\) implies that

\((*)\) \quad 1 - \Phi(z_n) = \left(2\pi\right)^{-1/2} z_n^{-1} \exp(-z_n^2/2) \{1 + o(1)\}

as \(n \to \infty\). Petrov's theorem then yields

Corollary 1.2. For \(1 < r < 2\) and \(\alpha = (2 - r)/r\), as \(n \to \infty\),

\[ P(S_n \geq \beta n^{1/r}) = \left(2\pi\beta^2 n^{\alpha} \right)^{-1/2} \exp\left\{-\frac{1}{2} \beta^2 n^{\alpha} (1 - \beta n^{(\alpha-1)/2} \lambda(\beta n^{(\alpha-1)/2})) \right\} \{1 + o(1)\}. \]

Proof. In Theorem 1.1 and (*) above, we take \(z_n = \beta n^{(2-r)/2r} = \beta n^{\alpha/2}\), where \(0 < \alpha < 1\). Therefore \(z_n^{-1} = \beta^{-1} n^{-\alpha/2}\), \(z_n^2 = \beta^2 n^{\alpha}\), \(z_n^3 = \beta^3 n^{3\alpha/2}\), and \(n^{-1/2}z_n = \beta n^{(\alpha-1)/2}\) as \(n \to \infty\). We can therefore write that

\[ P(S_n \geq \beta n^{1/r}) = P(n^{-1/2}S_n \geq \beta n^{\alpha/2}). \]

The result follows by inserting (*) into the conclusion of Theorem 1.1 and then making the substitutions for \(z_n\).

The following corollary expresses the large deviation result in the form needed for the proof of the main theorem:

Corollary 1.3. If \(\phi(t) < \infty\) for \(|t| < B\), where \(0 < B \leq \infty\), then for all \(\beta > 0\) and all sufficiently large \(n\), there exist numbers \(\theta_n\) depending on \(\lambda\) and \(\phi\) such that \(\theta_n \to 0\) as \(n \to \infty\) and

\[ \exp\left\{-\frac{1}{2} \beta^2 n^{\alpha} (1 + |\theta_n|)\right\} \leq 2(2\pi\beta^2 n^{\alpha})^{1/2} P(n^{-1/r}S_n \geq \beta) \leq 3 \exp\left\{-\frac{1}{2} \beta^2 n^{\alpha} (1 - |\theta_n|)\right\}, \]

where \(1 < r < 2\) and \(\alpha = (2 - r)/r\).
Proof. We apply Corollary 1.2, noting that $0 < \alpha < 1$ and that $\lambda(z)$ converges for all sufficiently small $z$. We then set
\[ \theta_n = \beta n^{(\alpha - 1)/2} \lambda(\beta n^{(\alpha-1)/2}), \]
and we take $n$ large enough so that $0.5 < 1 + o(1) < 1.5$.

2. The law of large numbers. This section contains the proof of the main theorem of this article, the extension of the Erdős-Rényi new law of large numbers. For $1 < r < 2$ and $S_n = \sum_{k=1}^{n} X_k$, we define
\[ \Sigma_r(N, K) = \max_{0 < n < N - K} \left| \frac{S_n}{n^{1/r}} \right| : 0 \leq n \leq N - K. \]

We now state and prove the main theorem:

Theorem 2.1. If $\{X_n : 1 \leq n < \infty\}$ is a sequence of i.i.d. random variables with $E(X_1) = 0$, $\text{Var}(X_1) = 1$, and $m.g.f. \phi(t) < \infty$ for $|t| < B$, where $0 < B \leq \infty$, and $1 < r < 2$, then, for every $\lambda > 0$,
\[ P\left\{ \lim_{N \to \infty} \Sigma_r(N, K^{(2\lambda - 2\log N)/a}) = \lambda \right\} = 1, \]
where $a = (2 - r)/r$.

Proof. We take $K_N = [(2\lambda - 2\log N)^{a}]$, and to simplify the notation, we set
\[ \Sigma_r(N, K) = \Sigma_r(N, K_N) \quad \text{and} \quad S_N(n, \lambda) = K_N^{-1/r}(S_n + K_N - S_n). \]

For $\epsilon > 0$ arbitrary, we define $\lambda'' = \lambda + \epsilon$, and we obtain from Corollary 1.3 that
\[ P(\Sigma_r(N, \lambda, N) \geq \lambda'') = P\left( \max_{0 \leq n \leq N - K_N} S_N(n, \lambda) \geq \lambda'' \right) \]
\[ \leq \sum_{n=0}^{N-K_N} P(S_N(n, \lambda) \geq \lambda'') \]
\[ \leq (N - K_N + 1)3(2\lambda'')^{-1}(2\pi K_N^a)^{-1/2} \exp\left\{ -\frac{1}{2}(\lambda'')^2 K_N^a \right\}(1 - |\theta''_N|), \]
where $\theta''_N = \theta_n(\lambda'')$. We can choose $N''$ so large that, for $N \geq N''$,
\[ \frac{1}{2}(\lambda'')^2 K_N^a (1 - |\theta''_N|) = \frac{1}{2}\lambda^2(1 + \delta_1) \exp\left\{ -\frac{1}{2}(\lambda'')^2 K_N^a (1 - |\theta''_N|) \right\} \]
\[ \geq \frac{1}{2}\lambda^2(1 + \delta_1) \lambda^{-2}\log N = (1 + \delta_1)\log N. \]
as \( K_N + 1 > (2\lambda^{-2}\log N)^{1/\alpha} \), where \( \delta_1 > 0 \) is a constant depending only on \( \lambda \) and \( \epsilon \). Increasing \( N" \), if necessary, so that also \( \{K_N(K_N + 1)^{1/\alpha} > \frac{1}{2} \) for \( N \geq N" \), we have that

\[
P(\Sigma(r, \lambda, N) \geq \lambda"') \leq 3N(2\lambda)^{-1}(4\pi\lambda^{-2}\log N)^{-1/2}\exp\{-(1 + \delta_1)\log N\}
\]

\[
\leq 3(\log N)^{-1/2}N^{-\delta_1} \leq 3N^{-\delta_1/2}.
\]

For all large values of \( N \) such that \( \lfloor(2\lambda^{-2}\log N)^{1/\alpha}\rfloor = j \), we have \( j \leq (2\lambda^{-2}\log N)^{1/\alpha} < j + 1 \), so that

\[
\exp\left\{\frac{1}{2}\lambda^2/j^2\right\} \leq N < \exp\left\{\frac{1}{2}\lambda^2/(j + 1)^2\right\}.
\]

We denote \( \exp\left(\frac{1}{2}\lambda^2/j^2\right) \) by \( E_j \), and then we define \( N_j \) to be the largest integer such that \( \lfloor(2\lambda^{-2}\log N)^{1/\alpha}\rfloor = j \), so that \( N_j \geq E_j \). It follows that, for \( j" = \lfloor(2\lambda^{-2}\log N)^{1/\alpha}\rfloor \),

\[
\sum_{j=j"}^{\infty} P(\Sigma_r(N_j, j) \geq \lambda") \leq \sum_{j=j"}^{\infty} 3N_j^{-\delta_1} \leq 3\sum_{j=j"}^{\infty} E_j^{-\delta_1} < \infty,
\]

since the series converges by the integral test. Therefore, by the Borel-Cantelli lemma, the probability is one that \( \Sigma_r(N_j, j) < \lambda" \) for all but finitely many values of \( j \). But, for each \( N \), such that \( N_{j-1} < N \leq N_j \), we have \( \Sigma_r(N, j) \leq \Sigma_r(N_j, j) \) because \( N \leq N_j \), so that \( \Sigma_r(N, j) \geq \lambda" \) implies \( \Sigma_r(N_j, j) \geq \lambda" \) as well. Therefore the probability is one that \( \Sigma_r(N, j) < \lambda" \) for all but finitely many values of \( j \), where \( j = K_N = \lfloor(2\lambda^{-2}\log N)^{1/\alpha}\rfloor \). It follows that

\[
P\left\{\limsup_{N \to \infty} \Sigma_r(N, [(2\lambda^{-2}\log N)^{1/\alpha}]) < \lambda + \epsilon\right\} = 1,
\]

and, since \( \epsilon > 0 \) is arbitrary, we have

\[
P\left\{\limsup_{N \to \infty} \Sigma_r(N, [(2\lambda^{-2}\log N)^{1/\alpha}]) \leq \lambda\right\} = 1.
\]

On the other hand, we take \( \epsilon > 0 \) arbitrary and define \( \lambda' = \lambda - \epsilon \). Then, using \( S_N[m, \lambda] \) to denote \( K_N^{-1/\alpha}(S_{(m+1)K_N} - S_{mK_N}) \) for integer values of \( m \), we have that

\[
P(\Sigma(r, \lambda, N) < \lambda') = P\left\{\max_{0 \leq n \leq N - K_N} S_N(n, \lambda) < \lambda'\right\}
\]

\[
\leq P\{S_N[m, \lambda] < \lambda' \text{ for } 0 \leq m \leq \lfloor N K_N^{-1}\rfloor - 1\}
\]

since \( mK_N = n \) and \( 0 \leq n \leq N - K_N \) imply that \( 0 \leq mK_N \leq N - K_N \), and

\((m + 1)K_N = N\) if and only if \( m = NK_N^{-1} - 1 \). By independence of the incre-
ments, we have
\[ P\{\sum_{n=0}^{[NK^{-1}]^{-1}} \leq \prod_{m=0}^{[NK^{-1}]^{-1}} P\{S_N[m, \lambda] < \lambda'\} \leq \{P(K_N^{-1/r} S_K < \lambda')\}^{[NK^{-1}]^{-1}}. \]

From Corollary 1.3, we can write that
\[ P\{K_N^{-1/r} S_K < \lambda'\} = 1 - P\{S_K \geq \lambda' K_N^{1/r}\} \]
\[ \leq 1 - (2\lambda')^{-1}(2\pi K_N^{\alpha})^{-1/2} \exp\{-\frac{1}{2}(\lambda')^2 K_N^\alpha (1 + |\theta'_{K_N}|)\}, \]
where \( \theta'_n = \theta_n (\lambda') \). Now \( \lambda' = \lambda - \epsilon \) and \( K_N \leq (2\lambda^{-2} \log N)^{1/\alpha} \) imply that
\[ \frac{1}{2}(\lambda')^2 K_N^\alpha (1 + |\theta'_{K_N}|) \leq \frac{1}{2} \lambda^2 (1 - \epsilon \lambda^{-1})^2 (2\lambda^{-2} \log N) (1 + |\theta'_{K_N}|) \]
\[ \leq (1 - \epsilon \lambda^{-1}) (\log N) (1 + |\theta'_{K_N}|). \]

We can choose \( N' \) so large that \((1 - \epsilon \lambda^{-1}) (1 + |\theta'_{K_N}|) < 1 - 3\delta_2\), where \( \delta_2 > 0 \) is a constant depending only on \( \lambda \) and \( \epsilon \). Therefore
\[ P\{K_N^{-1/r} S_K < \lambda'\} \leq 1 - (2\lambda)^{-1}(2\pi 2\lambda^{-2} \log N)^{-1/2} \exp\{-\frac{1}{2}(\lambda')^2 K_N^{-1} (1 + |\theta'_{K_N}|)\}, \]
\[ \leq 1 - (16\pi \log N)^{-1/2} \exp\{-\frac{1}{2}(1 - 2\delta_2)\}, \]
\[ \leq 1 - N^{-1} \exp\{-N^{-1} - 2\delta_2\}, \]

taking \( N' \) also large enough so that \((16\pi \log N)^{-1/2} \leq N^{-\delta_2}\). It follows that, for all \( N \geq N'\),
\[ P\{\sum_{n=0}^{[NK^{-1}]^{-1}} \leq \exp\{-[NK^{-1}] N^{-\delta_2}\} \leq \exp\{-N^{-\delta_2}\}, \]
if we take \( N' \) large enough so that \( K_N \leq N^{\delta_2} \) and \( [NK^{-1}] \geq N^{1-\delta_2} \). Then
\[ \sum_{N=N'}^{\infty} P\{\sum_{n=0}^{[NK^{-1}]^{-1}} \leq \exp\{-N^{-\delta_2}\} < \infty \]
by the integral test. Therefore, the Borel-Cantelli lemma asserts that, with probability one, only finitely many \( \sum_{n=0}^{[NK^{-1}]^{-1}} \) are less than \( \lambda' \). Therefore
\[ P\{\liminf_{N \to \infty} \Sigma(r, \lambda, N) \geq \lambda'\} = 1. \]

Since \( \lambda' = \lambda - \epsilon \), where \( \epsilon > 0 \) is arbitrary, this means that
\[ P\{\liminf_{N \to \infty} \Sigma(r, N, [(2\lambda^{-2} \log N)^{1/\alpha}]) \geq \lambda\} = 1, \]
and the theorem follows.
3. Comparison with the Erdös-Rényi theorem. The Erdös-Rényi theorem makes the following assertion:

**Theorem 3.1 (Erdös-Rényi).** If \( \{X_n: 1 \leq n < \infty\} \) is a sequence of i.i.d. random variables with \( E(X_1) = 0 \), \( \text{Var}(X_1) = 1 \), and m.g.f. \( \phi(t) < \infty \) for a \( |t| < B \), where \( 0 < B \leq \infty \), then, for every \( \lambda \) in the range of \( Q(t) = \phi'(t)/\phi(t) \),

\[
P\left\{ \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \left[\frac{(C(\lambda) \log N)}{(\lambda^2 \log(N^2) + C(\lambda) \log N)}\right] = \lambda \right\} = 1,
\]

where \( C(\lambda) = (\lambda^2 \log(N^2) + \log \phi(Q^{-1}(\lambda)))^{-1} \).

We note that in going from Theorem 2.1 (which holds only for \( 1 < r < 2 \)) to Theorem 3.1 (which holds only for \( r = 1 \)), the quantity \( 2\lambda^{-2} \) suddenly jumps to \( C(\lambda) \). Why does that instantaneous jump occur? We observe that, by L'Hôpital’s Rule,

\[
\lim_{\lambda \to 0} 2\lambda^{-2}/C(\lambda) = 2 \lim_{\lambda \to 0} \lambda^{-2} \left(\lambda^2 \log(N^2) + \log \phi(Q^{-1}(\lambda))\right) = \lim_{\lambda \to 0} \lambda^{-1}Q^{-1}(\lambda) = (Q^{-1})'(0) = 1,
\]

as \( \text{Var}(X_1) = 1 \). This seems to say that as \( \lambda n \) gets closer to \( \lambda n^{1/r} \) (by the process of \( \lambda \) tending to \( 0 \)), \( C(\lambda) \) gets closer to \( 2\lambda^{-2} \), which would be its actual value if \( \lambda n \) really were \( \lambda n^{1/r} \) for \( 1 < r < 2 \).

In the special case of normally distributed \( X_1 \), it turns out that \( C(\lambda) = 2\lambda^{-2} \) for all values of \( \lambda \), because \( Q(t) = t \) in that case. We have therefore that

\[
P\left\{ \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \left[\frac{(2\lambda^{-2} \log N)^{1/2}}{(\lambda^2 \log(N^2) + 2\lambda^{-2} \log N)}\right] = \lambda \right\} = 1
\]

for \( 1 \leq r < 2 \) in the normal case, where \( \alpha = (2 - r)/r \). In the general case, however, for \( r = 1 \), denoting the value of \( C(\lambda) \) in Theorem 3.1 by \( 2\theta^{-2} \) so that \( \lambda \) is \( C^{-1}(2\theta^{-2}) \), we can write

\[
P\left\{ \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \left[\frac{(2\theta^{-2} \log N)^{1/2}}{(\lambda^2 \log(N^2) + 2\theta^{-2} \log N)}\right] = C^{-1}(2\theta^{-2}) \right\} = 1,
\]

where \( x = C^{-1}(2\theta^{-2}) \) is the (unique) solution of the equation

\[
xQ^{-1}(x) - \log \phi(Q^{-1}(x)) = \theta^2/2.
\]

The solution exists if \( C^{-1}(2\theta^{-2}) \) lies in the range of \( Q \). Now, because \( C(\lambda) \) is a strictly decreasing function of \( \lambda \), and \( C(\lambda) < 2\lambda^{-2} \) for \( \lambda > 0 \), it follows that \( C^{-1}(2\lambda^{-2}) > C^{-1}(C(\lambda)) = \lambda \). Under the conditions of Theorem 2.1, we then have with probability one that

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\[
\lim_{N \to \infty} \sum_{r=1}^{N} [(2\lambda^{-2} \log N)^{1/\alpha}] = \lambda \quad \text{for } 1 < r < 2,
\]
and
\[
\lim_{N \to \infty} \sum_{r=1}^{N} (2\lambda^{-2} \log N) = C^{-1}(2\lambda^{-2}) \quad \text{for } r = 1,
\]
where, in the case of normality, \(C^{-1}(2\lambda^{-2}) = \lambda\).

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REFERENCES