ON THE PRODUCTS OF WEAKLY LINDELOF SPACES

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ABSTRACT. The aim of this note is to show, without using any special set-theoretic assumptions, that the product of two (weakly) Lindelöf spaces is not necessarily weakly Lindelöf.

In [1] M. Ulmer has constructed two weakly Lindelöf spaces whose product is not so; in his construction, the assumption \( 2^{\aleph_0} = 2^{\aleph_1} \) was essentially employed. In this short note we shall provide another such example (where the factors are even Lindelöf), in the construction of which no additional set-theoretic assumption is used.

To start with, we shall deal with some properties of the "half-open" topologies on linearly ordered sets which may be interesting in themselves. We recall that, given a cardinal number \( \alpha \), \( X \) is (weakly) \( \alpha \)-Lindelöf if every open cover of \( X \) has a subcover (weak subcover, i.e. a subfamily whose union is dense in \( X \)) of cardinality \( \leq \alpha \).

Let \( (\mathbb{R}, <) \) be a linearly ordered set. We shall denote by \( R^+ \) and \( R^- \), respectively, the spaces on \( R \) for which the half-open intervals of the form \( [x, y) \) and \( (x, y] \), respectively, form an open basis.

Lemma 1. Let \( \alpha \) be an infinite cardinal number, and let \( \langle R, < \rangle \) be an order complete linearly ordered set in which there is no decreasingly or increasingly ordered subset of type \( \alpha^+ \). Then both \( R^+ \) and \( R^- \) are \( \alpha \)-Lindelöf spaces.

Proof. It will obviously suffice to show that \( R^+ \) is \( \alpha \)-Lindelöf. To see this, let \( \mathcal{U} \) be a cover of \( R^+ \) by basic open sets of the form \( [x, y) \). First we claim that for any \( a, b \in R, a < b \), the segment \( [a, b] \) can be covered by at most \( \alpha \) members of \( \mathcal{U} \).

Indeed, using the completeness of \( \langle R, < \rangle \), there is a \( c \in [a, b] \) which is the least upper bound of those \( d \in [a, b] \) for which the segment \( [a, d] \) can be covered by \( \leq \alpha \) members of \( \mathcal{U} \). We claim that \( c = b \).

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Suppose, on the contrary, that \( c < b \). The segment \([a, c]\) itself can be covered by \( \leq \alpha \) members of \( \mathcal{U} \). This is obvious if \( c = a \) or if \( c \) has an immediate predecessor. If not, it follows from the second condition on \( (\mathbb{R}, -<) \).

Indeed, in this case we can take an increasing, well-ordered sequence \( \langle c_\xi, \xi < \lambda \rangle \) converging to \( c \) from below, with \( c_\xi \in (a, c) \) and \( \lambda \leq \alpha \). By the choice of \( c \), every segment \([a, c_\xi]\) can be covered by \( \leq \alpha \) members of \( \mathcal{U} \); hence, so can \([a, c) = \bigcup\{[a, c_\xi]: \xi < \lambda\}\), and \([a, c]\) as well.

Now there are two cases to be distinguished.

First, if \( c \) has an immediate successor, say \( c' \), and \([x, y)\) is a member of \( \mathcal{U} \) containing \( c' \), then adding \([x, y)\) to any cover of \([a, c]\) with \( \leq \alpha \) members of \( \mathcal{U} \) we obtain such a cover of \([a, c']\), contradicting the choice of \( c \). Similarly if \( c \) has no immediate successor and \([x, y)\) is a member of \( \mathcal{U} \) containing \( c \), then \([x, y)\) contains a \( c' > c \), hence, adding it to an appropriate cover of \([a, c]\), we again get a contradiction.

Now making use again of the second condition on \( (\mathbb{R}, -<) \), we can obtain sequences \( \langle a_\xi: \xi < \alpha \rangle \) and \( \langle b_\xi: \xi < \alpha \rangle \) such that the first one is coinitial and the second is cofinal in \( (\mathbb{R}, -<) \). According to what we have proved above, every segment \([a_\xi, b_\eta]\) can be covered by \( \leq \alpha \) members of \( \mathcal{U} \), hence so can

\[
\mathbb{R}^+ = \bigcup\{[a_\xi, b_\eta]: \xi, \eta < \alpha\}.
\]

This completes the proof.

**Lemma 2.** Let \( (\mathbb{R}, -<) \) be a densely ordered set with \( d(\mathbb{R}) > \beta \) (i.e., \( \mathbb{R} \) does not contain a dense subset of cardinality \( \leq \beta \)). Then the product space \( \mathbb{R}^+ \times \mathbb{R}^- \) is not weakly \( \beta \)-Lindelöf.

**Proof.** Let us denote, as usual, by \( \Delta \) the diagonal \( \Delta = \{(p, p): p \in \mathbb{R}\} \) of the product \( \mathbb{R}^+ \times \mathbb{R}^- \), and put

\[
\Gamma = \{(p, q) \in \mathbb{R}^+ \times \mathbb{R}^-, p < q\}.
\]

First we show that \( \Gamma \) is open in \( \mathbb{R}^+ \times \mathbb{R}^- \). Indeed if \( p < q \) then, by the denseness of \( (\mathbb{R}, -<) \), there is an \( r \) with \( p < r < q \), and obviously \([p, r) \times (r, q)\) is a neighbourhood of \((p, q)\) contained in \( \Gamma \).

Thus

\[
\mathcal{U} = \{\Gamma\} \cup \{[p, -\rightarrow) \times (-\leftarrow, p]: p \in \mathbb{R}\}
\]

is an open cover of \( \mathbb{R}^+ \times \mathbb{R}^- \), since for any \((p, q)\) with \( p \geq q \) we have

\[
(p, q) \in [p, -\rightarrow) \times (-\leftarrow, p].
\]

We claim that for any subfamily \( \mathcal{V} \subset \mathcal{U} \) with \( |\mathcal{V}| = \beta \), the union \( \mathcal{V} = \bigcup \mathcal{V} \) is...
not dense in \( R^+ \times R^- \). Indeed, let us put

\[ A = \{ p \in R : [p, \rightarrow) \times (\leftarrow, p] \in \mathcal{O} \}. \]

Then \( |A| \leq |\mathcal{O}| = \beta < d(R) \), so there is an open interval \((a, b)\) of \( R \) such that \( A \cap (a, b) = \emptyset \); and by the denseness of \( \langle R, \langle \rangle \rangle \), there is a \( c \) with \( a < c < b \).

Now the set \([c, b) \times (a, c]\) is obviously a nonempty open subset of \( R^+ \times R^- \), and for any \( p \in A \) we have

\[ [p, \rightarrow) \times (\leftarrow, p] \cap [c, b) \times (a, c] = \emptyset, \]

since either \( p \geq b \) or \( p \leq a \). Moreover, we have, trivially,

\[ V \cap [c, b) \times (a, c] = \emptyset. \]

This indeed shows

\[ V \cap [c, b) \times (a, c] = \emptyset, \]

so that \( V \) is not dense; thus \( R^+ \times R^- \) has an open cover with no weak sub-cover of cardinality \( \leq \beta \), and consequently \( R^+ \times R^- \) is not weakly \( \beta \)-Lindelöf.

Now we are ready to present our Example.

Example. Let \( \langle R, \langle \rangle \rangle \) be any linearly ordered set satisfying the following conditions:

(i) \( \langle R, \langle \rangle \rangle \) is continuously (i.e. both densely and completely) ordered;
(ii) \( \langle R, \langle \rangle \rangle \) contains no uncountable decreasing or increasing well-ordered subset;
(iii) \( d(R) = 2^{\aleph_0} \).

(The unit square with the lexicographic ordering is such an ordered set.) Then \( R^+ \) and \( R^- \) are Lindelöf spaces such that \( R^+ \times R^- \) is not weakly Lindelöf, nor even weakly \( \beta \)-Lindelöf for any \( \beta < 2^{\aleph_0} \).

The proof is obvious from Lemmas 1 and 2.

REFERENCE


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