MORSE-SMALE ENDOMORPHISMS OF THE CIRCLE

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ABSTRACT. The orbit structure of a continuously differentiable map $f$ of the circle is examined, in the case where the nonwandering set of $f$ is finite and hyperbolic. It is shown that there is a natural number $n(f)$ such that the period of any periodic point of $f$ is $n(f)$ times a power of 2.

1. Introduction. It is well known (see [4]) that for a Kupka-Smale diffeomorphism $f$ of the circle $S^1$ with $\Omega(f)$ finite, the following are true:

A. $\Omega(f)$ consists of periodic points.

B. The expanding and contracting periodic points alternate.

C. If $f$ is orientation preserving, all periodic points have the same period, and if $f$ is orientation reversing all periodic points have period one or two.

The purpose of this paper is to determine to what extent A, B, and C are true for a Kupka-Smale endomorphism $f$ of $S^1$ with $\Omega(f)$ finite. (To avoid unnecessary confusion caused by certain pathological cases, we also assume a generic property about the singularities of $f$.) The results are stated in Theorems A, B, and C, following the necessary definitions.

We let $\text{End}(S^1)$ denote the space of $C^1$ maps of $S^1$ into itself. Fix $f \in \text{End}(S^1)$. A point $x \in S^1$ is said to be wandering if there is a neighborhood $N$ of $x$ in $S^1$ such that $f^i(N) \cap N = \emptyset$, $\forall i > 0$. The set of points which are not wandering is called the nonwandering set and denoted $\Omega(f)$. A point $x \in S^1$ is called a periodic point if $f^n(x) = x$ for some natural number $n$. The minimum of $|n|/f^n(x) = x|$ is called the period of $x$.

A periodic point $x$ of period $n$ is said to be contracting if $|Df^n(x)| < 1$, and expanding if $|Df^n(x)| > 1$. $f$ is said to be Kupka-Smale if any periodic point of $f$ is expanding or contracting.

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1 This paper is a revision of a part of the author's thesis [1] completed at Northwestern University (and the Institut des Hautes Études Scientifiques) under the supervision of R. F. Williams.
A point \( x \in S^1 \) is called a singularity of \( f \) if \( Df(x) = 0 \). \( x \) is said to be eventually periodic if \( f^m(x) \) is periodic for some natural number \( m \), or equivalently if \( \text{orb}(x) \) is finite, where \( \text{orb}(x) = \{ f^n(x) \mid n \geq 0 \} \).

We now define \( MS(S^1) \) to be the set of \( f \in \text{End}(S^1) \) such that \( \Omega(f) \) is finite, and:

1. \( f \) is Kupka-Smale.
2. No singularity of \( f \) is eventually periodic.

For \( f \in MS(S^1) \) we let \( \Omega_c(f) \) (respectively \( \Omega_e(f) \)) denote the set of contracting (respectively expanding) periodic points of \( f \).

We will prove the following:

**Theorem A.** If \( f \in MS(S^1) \) then \( \Omega(f) \) consists of periodic points.

**Theorem B.** Let \( f \in MS(S^1) \), and \( \text{card} \) denote cardinality.
\[
\text{card} \Omega_c(f) \leq \text{card} \Omega_e(f) \leq \text{card} \Omega_c(f) + 1.
\]

Equality (of \( \text{card} \Omega_e(f) \) and \( \text{card} \Omega_c(f) \)) holds if and only if \( f \) is onto. In the onto case the expanding and contracting periodic points alternate.

**Theorem C.** Let \( f \in MS(S^1) \). There is a natural number \( n(f) \) such that the period of any periodic point of \( f \) is \( n(f) \) times a power of 2. (Here we include \( 1 = 2^0 \) as a power of 2.)

We conclude this section with a few remarks. First, suppose \( f \in MS(S^1) \) is \( C^2 \) and satisfies the additional generic properties:

3. All singularities of \( f \) are nondegenerate (i.e. the second derivative is not zero).

4. Orbits of distinct turning points are disjoint.

Then \( f \) is structurally stable (see [1] or [3]). In fact, the set of maps \( f \) satisfying these properties can be classified up to topological conjugacy, by associating to each such \( f \) a finite diagram consisting of certain eventually periodic points of \( f \) and iterates of the singularities of \( f \) (see [1] for details, or [2] where a special case is studied).

Second, since \( x \in \Omega(f) \Rightarrow f(x) \in \Omega(f) \), it is obvious that \( f \in MS(S^1) \) and \( x \in \Omega(f) \) imply \( \text{orb}(x) \) is finite. However this does not mean \( x \) is periodic for endomorphisms. So Theorem A is not immediate as it is in the diffeomorphism case.

Third, we note that one can construct (by induction) for any natural number \( n \), a map \( f_n \) in \( MS(S^1) \) with periodic points of period 1, 2, 4, \( \cdots \), \( 2^n \) (see [1] for details). Thus the statement in Theorem C is essentially the most that can be said.
Finally, we remark that Theorems A, B, and C are true without the assumption that no singularity is eventually periodic. However, dropping this assumption makes a few of the proofs somewhat cumbersome, while adding little generality.

2. Proof of Theorem A. Let $f \in \text{End}(S^1)$, and let $x$ be an expanding periodic point of period $n$. We let $W_f^u(x)$ denote the local unstable manifold of $x$, which is simply an open interval about $x$ on which $|Df^n(x)| > 1$, such that $f^n(W_f^u(x)) \supset W_f^u(x)$. We set $W_f^u(x) = \text{orb}(W_f^u(x))$, where $\text{orb}(A)$ is defined for any set $A$ by $\text{orb}(A) = \bigcup_{n \geq 0} f^n(A)$.

We will use the following remark in the proof of Proposition 1. If $g$ is a continuous map of $S^1$ into itself, and $I$ is a closed interval in $S^1$ with $g(I) \supset I$ and $g(I) \not\subset S^1$, then $g$ has a fixed point in $I$. This statement follows immediately from continuity (Rolle’s theorem), but is false without the hypothesis $g(I) \subset S^1$.

Theorem A follows immediately from the following proposition.

**Proposition 1.** Suppose $f \in \text{End}(S^1)$ is Kupka-Smale and no singularity of $f$ is eventually periodic. Suppose $y \in \Omega(f)$ is eventually periodic but not periodic. Then $y$ is a limit of periodic points.

**Proof.** By hypothesis there is an expanding periodic point $p$ and an integer $k > 0$ with $f^k(y) = p$. Let $V$ be any neighborhood of $y$. By choosing $V$ smaller if necessary, we may assume that $f^k(V)$ is a neighborhood of $p$ in $W_f^u(p)$.

Note that $y \in W_f^u(p)$ or else $y$ would be wandering. But since $W_f^u(p) - W_f^u(p)$ is a finite invariant set, we have $y \in W_f^u(p)$. Hence $\exists y_1 \in W_f^u(p)$ and $n > 0$ with $f^n(y_1) = y$. Let $W$ be a closed interval about $y_1$ in $W_f^u(p)$ such that $f^n(W)$ is a neighborhood of $y$ in $V$. Then $f^{n+k}(W)$ is a neighborhood of $p$ in $W_f^u(p)$.

Now, there is a closed interval $K \subset f^{n+k}(W)$, and an integer $l > 0$, such that $f^l(K) = W$. So, $f^{n+k+l}(K) = f^{n+k}(W)$, which is a proper closed interval containing $K$. Hence $K$ contains a periodic point, which implies that all iterates of $K$ contain periodic points. In particular, since $V \supset f^n(W) = f^{n+l}(K)$, $V$ has a periodic point. Since $V$ was arbitrary this completes the proof. Q.E.D.

3. Proof of Theorem B.

**Lemma 2.** Let $f \in \text{MS}(S^1)$ and let $p$ be an expanding periodic point of $f$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
There does not exist \( y \in (W^u(p) - \text{orb}(p)) \) with \( p \in \text{orb}(y) \).

**Proof.** Such an element \( y \) would be nonwandering, but not periodic, a contradiction by Theorem A. Q.E.D.

We now make another definition. We will use the notation \([a, b]\) to denote the arc from \( a \) to \( b \) in which \( b \) is in the counterclockwise direction from \( a \). Let \( f \in MS(S^1) \). Let \( p \) be an orientation preserving expanding fixed point of \( f \). Set \( W^u(p, cc) = \text{orb}[p, b] \), where \( b \) is a point in \( W^u_1(p) \) in the counterclockwise direction from \( p \), and set \( W^u(p, cl) = \text{orb}[a, p] \), where \( a \) is a point in \( W^u_1(p) \) in the clockwise direction from \( p \). From the definition of \( W^u_1(p) \), it follows that \( W^u(p, cc) \) and \( W^u(p, cl) \) are independent of the choices for \( a \) and \( b \). If \( p \) is an orientation reversing expanding fixed point, define \( W^u(p, cc) \) and \( W^u(p, cl) \) by thinking of \( p \) as an orientation preserving fixed point of \( f^2 \). Finally, if \( p \) is an expanding periodic point of period \( n \), define \( W^u(p, cc) \) and \( W^u(p, cl) \) by thinking of \( p \) as a fixed point of \( f^n \).

**Proposition 3.** Let \( p \) be an expanding periodic point of \( f \in MS(S^1) \) and let \( I = W^u(p, cc) \) or \( I = W^u(p, cl) \). Then \( I \) is a proper subinterval of \( S^1 \) which contains another periodic point (besides \( p \)), and the closest periodic point to \( p \) in \( I \) is contracting.

**Proof.** By looking at a power of \( f \), we may assume without loss of generality that \( p \) is an orientation preserving fixed point. We may also assume that \( I = W^u(p, cc) \). If \( I = S^1 \), \( \exists y \neq p \) in \( W^u(p, cc) \) with \( f(y) = p \). This contradicts Lemma 2. Hence \( I \) is a proper subinterval of \( S^1 \). Let \( I = [p, b] \).

We put an order \( < \) on \( I \) by identifying \( I \) with a subinterval of the real line. If \( f(b) = b \) then \( b \) is a fixed point of \( f \) in \( I \). If not \( f(b) < b \). Since \( p \) is expanding, \( \exists d \) in \( W^u_I(p) \) in \([p, b]\) with \( d < f(d) \). By continuity \( f \) has a fixed point in \([p, b]\).

Let \( c \) be the closest periodic point to \( p \) in \( I \). We must show that \( c \) is contracting. Without loss of generality we may assume that \( c \) is an orientation preserving fixed point. Suppose \( c \) is expanding. \( \exists l < c \), with \( f(l) < l \). Hence there is a fixed point in \([d, l]\). This contradicts the fact that \( c \) is the closest periodic point to \( p \) in \( I \). Hence \( c \) is contracting. Q.E.D.

If \( c \) is a contracting periodic point of period \( n \) of \( f \in \text{End}(S^1) \), we define the stable manifold of \( c \) by \( W^s(c) = \{ x \in S^1 | c \text{ is a limit point of } \text{orb}(x) \} \). The component of \( W^s(c) \) which contains \( c \) is called the semilocal stable manifold of \( c \), and is denoted by \( \text{slsm}(c) \).

**Proposition 4.** Let \( c \) be a contracting periodic point of \( f \in MS(S^1) \). If
slsm(c) \neq S^1$, then one of the endpoints of slsm(c) is an expanding periodic point.

Proof. Let $E$ be the set of endpoints of slsm(c). $E$ has one or two elements and $f^n(E) \subseteq E$, where $c$ is of period $n$. Hence $f$ has a periodic point in $E$. We show that any periodic point $p \in E$ is expanding. Suppose $p$ is contracting. We may assume that $c$ and $p$ are orientation preserving fixed points, and $p$ is in the counterclockwise direction from $c$. Put an order $< \leq [c, p]$ as in Proposition 3. \exists a and $b$ in $(c, p)$ with $f(a) < a$ and $b < f(b)$. Hence there is a fixed point in $(c, p)$. This is a contradiction since $c$ is the only fixed point in slsm(c). Q.E.D.

The following proposition follows almost immediately from the Lefschetz trace formula (see [6]).

Proposition 5. If $f \in MS(S^1)$ then the degree of $f$ is $0, +1$, or $-1$. If the degree of $f$ is $0$, then $\card \Omega_f = \card \Omega_e(f) + 1$. If the degree of $f$ is $\pm 1$ then $\card \Omega_f = \card \Omega_e(f)$.

Proposition 6. Let $f \in MS(S^1)$ be onto. Then $\card \Omega_e(f) = \card \Omega_f$.

Proof. Without loss of generality we may assume that all the periodic points of $f$ are orientation preserving fixed points. Suppose the statement is false. Then there are two contracting fixed points $c_1$ and $c_2$ such that the interval $(c_1, c_2)$ contains no fixed points. (The only other possibility is that $\Omega(f)$ consists of a single fixed sink $c$, but this would imply $f$ is not onto by Proposition 4.)

Let $I = [c_1, c_2]$. Pick points $t_1 \in \text{slsm}(c_1)$ and $t_2 \in \text{slsm}(c_2)$ in $I$, such that $f(t_1) \in (c_1, t_1)$ and $f(t_2) \in (t_2, c_2)$. Let $J = [t_1, t_2]$. Then $f(J) \supseteq [f(t_2), f(t_1)]$. (For, if $f(J)$ did not contain this interval, it would have to contain $[f(t_1), f(t_2)]$. Then $f(J) \supseteq J$ and $f(J)$ is a proper subinterval of $S^1$. Hence there is a fixed point in $J$, a contradiction.)

Let $\Omega_f = \{e_1, \ldots, e_n\}$. There are points $k_1, \ldots, k_n$ in $J$ such that $f(k_i) = e_i$ for $i = 1, \ldots, n$. Since $f$ is onto for each $i = 1, \ldots, n$, we can find a sequence $(k_i^{-m})$ with $k_i^0 = k_i$ and $f(k_i^{-m}) = k_i^{-m+1}$ for $m > 0$. The sequence $(k_i^{-m})$ must have a limit point, and a limit point of this sequence is clearly nonwandering. So to each $k_i$ we can assign an expanding fixed point $e_j$ such that $e_j$ is a limit point of the sequence $(k_i^{-m})$. Define a map $T: \{k_1, k_2, \ldots, k_n\} \rightarrow \{k_1, k_2, \ldots, k_n\}$ by $T(k_i) = k_{j_i}$, where $e_j$ is the chosen limit point of $(k_i^{-m})$. Any map from a finite set into itself has a periodic
point, so there is a subset of \( \{k_1, k_2, \ldots, k_n\} \), say \( \{k_{j_1}, \ldots, k_{j_r}\} \), such that 

\[ T(j_i, k) = k_{j_i + 1} \text{ for } i = 1, \ldots, r - 1 \quad \text{and} \quad T(j_r, k) = k_{j_1}. \]

Let \( U \) be any neighborhood of \( k_{j_1} \). \( T(k_{j_r}) = k_{j_1} \) means that \( k_{j_1} \) is a limit point of \( (k_{j_r - m}^{-m}) \). Now \( f(U) \) is a neighborhood of \( e_{j_1} \) so some iterate of \( U \) contains \( k_{j_1} \). Then \( T(k_{j_r}) = k_{j_1} \) means \( e_{j_1} \) is a limit point of \( (k_{j_r}^{-r}) \). So some iterate of \( U \) contains \( k_{j_1} \). It follows after \( r - 2 \) more steps that an iterate of \( U \) contains \( k_{j_1} \) and hence intersects \( U \). Since \( U \) was arbitrary, \( k_{j_1} \) is nonwandering. This is a contradiction and completes the proof. Q.E.D.

Theorem B now follows from Propositions 5 and 6 (and the fact that if \( f \) is not onto then the degree of \( f \) is 0). In view of Proposition 5, the following corollary is essentially a restatement of the content of Theorem B.

**Corollary 7.** If \( f \in MS(S^1) \) and the degree of \( f \) is 0 then \( f \) is not onto.

4. Proof of Theorem C.

**Proposition 8.** Let \( e \) and \( c \) be adjacent expanding and contracting periodic points of \( f \in MS(S^1) \) with \( c \) fixed. Then \( e \) is fixed by \( f^2 \).

**Proof.** Without loss of generality we may assume that there are no periodic points in \((e, c)\). Let \( e_1 \) be the closest point to \( e \) in the counterclockwise direction from \( e \), in \( orb(e) \). We have two cases.

**Case 1.** \( f(e) \neq e_1 \). Then \( f([e, c]) \) contains \( c \) and the point \( f(e) \) which is not in \([e, e_1]\). Hence \( \exists x \in (e, c) \) such that \( f(x) = e \) or \( f(x) = e_1 \). In either case \( e \in orb(x) \), a contradiction by Lemma 2 and Proposition 3.

**Case 2.** \( f(e) = e_1 \). Note \([e, e_1] \subset W^u(e) \), because \( f([e, c]) \) is an interval containing \( c \) and \( e_1 \), so \( f([e, c]) \supset [c, e_1] \). If \( f(e_1) = e \) we are done, so suppose \( f(e_1) \neq e \). Then \( f([e, e_1]) \) contains \( c \), and the point \( f(e_1) \) is not in \([e, e_1]\). Hence \( \exists y \in (e, e_1) \) such that \( f(y) = e \) or \( f(y) = e_1 \). In either case \( e \in orb(y) \), a contradiction by Lemma 2.

**Proposition 9.** Let \( e \) and \( c \) be adjacent expanding and contracting periodic points of \( f \in MS(S^1) \) with \( e \) fixed. Then \( c \) has period a power of 2. (Here we include \( 1 = 2^0 \) as a power of 2.)

**Proof.** Suppose not. Without loss of generality we may assume that there are no periodic points in \((e, c)\). Let \( p \) be the closest periodic point to \( e \), in the counterclockwise direction from \( e \), which has period a power of 2 (there is such a \( p \) by the proof of Proposition 3). Suppose \( p \) is of period
If we let $g = f^{2k}$, then in the interval $[e, p]$, $g$ has only two fixed points, $e$ and $p$, both of which are orientation preserving. It follows that $p$ is contracting. For if $p$ is expanding, then by the proof of Proposition 3, $p \in W^u(e, cc)$ and $e \in W^u(p, cl)$. This implies that there is a nonperiodic nonwandering point in $W^u(e, cc)$, a contradiction.

Let $b$ be the closest periodic point to $p$ in $(e, p)$. Then $b$ is expanding by the proof of Theorem B, since $[b, p]$ is in $\text{Im}(g)$. Under $g$, $p$ is a contracting fixed point, and $b$ is an expanding periodic point of period greater than 2. This contradicts Proposition 8. Q.E.D.

Theorem C now follows from Propositions 8 and 9 and Theorem B.

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