UNIQUE BEST NONLINEAR APPROXIMATION
IN HILBERT SPACES

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ABSTRACT. Using the notion of curvature of a manifold, developed
by J. R. Rice and recently studied by E. R. Rozema and the second named
author, the authors prove the following result: Let $H$ be a Hilbert space
and $F$ map $\mathbb{R}^n$ into $H$ such that $F$ is a homeomorphism onto $\mathcal{F} = F(\mathbb{R}^n)$
and is twice continuously Fréchet differentiable. Then if $F'(a) \cdot \mathbb{R}^n$ is of
dimension $n$ for all $a \in \mathbb{R}^n$, the manifold $\mathcal{F}$ has finite curvature every-
where. It follows that there is a neighborhood $\mathcal{U}$ of $\mathcal{F}$ such that each $u \in$
$\mathcal{U}$ has a unique best approximation from $\mathcal{F}$. However, these results do not
hold in general for uniformly smooth Banach spaces.

1. Introduction. Nonlinear approximation arises naturally in many prob-
lems. In particular, where algorithmic solutions are desired, uniqueness
questions are of great concern. Many recent papers have been written on this
subject; for example, see [3], [4], [5], [6], and [7]. In [4], Rozema and the
second named author have studied the question of unique best approximation
in uniformly smooth Banach spaces. This note is a completion of [4] in the
sense that for a Hilbert space it gives a sufficient condition for finite curva-
ture, and hence unique best approximation, in terms of smoothness of the map.

Let $F$ map $\mathbb{R}^n$ into a Hilbert space $H$. We will denote by $\mathcal{F}$ the image
of $\mathbb{R}^n$ under $F$ and by $F'(a)$ the Fréchet derivative of $F$ at $a \in \mathbb{R}^n$.
Throughout this paper we will assume that $F$ satisfies the following three
conditions:

(a) $F$ is a homeomorphism onto its image $\mathcal{F}$ (where $\mathcal{F}$ is given the
topology induced by $H$).

(b) The first and second Fréchet derivatives of $F$ exist and are conti-
uous. (We will denote the second Fréchet derivative of $F$ at $a$ by $F''(a)$.)

(c) The dimension of $F'(a) \cdot \mathbb{R}^n$ is $n$ for each $a \in \mathbb{R}^n$.

As in [4], we define the tangent plane $T(x)$ to $\mathcal{F}$ at $x = F(a)$ by $T(x)$
$= x + F'(a) \cdot \mathbb{R}^n$, and the set of normals $\mathcal{N}(x)$ to $\mathcal{F}$ at $x$ by $\mathcal{N}(x) =$

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\{y: (y - x) \perp T(x)\}$. Let $x$ and $z$ be in $\mathcal{F}$, and for $y \in \Pi(x)$, we set
\[
\rho(x, y, z) = \inf \|y(z) - z\|
\]
where the infimum is taken over all the $y(z)$ that lie on the line generated by $x$ and $y$ and satisfy $\|y(z) - x\| = \|y(z) - z\|$. If no $y(z)$ exists, we set $\rho(x, y, z) = \infty$. Let
\[
\rho(x, y) = \lim_{z \to x, z \in \mathcal{F}} \inf \rho(x, y, z).
\]
The radius of curvature $\rho(x)$ of $\mathcal{F}$ at $x$ is defined in [4] as
\[
\rho(x) = \inf_{y \in \Pi(x)} \rho(x, y)
\]
and the curvature $\sigma(x)$ of $\mathcal{F}$ at $x$ is defined by $\sigma(x) = 1/\rho(x)$. For simplicity we will only consider real Hilbert spaces.

2. Finite curvature in terms of smoothness. We first establish the following theorem for a Hilbert space $H$. In the next section we will see that this result does not necessarily hold for a uniformly smooth Banach space in general.

**Theorem 1.** Let $F: \mathbb{R}^n \to H$ satisfy (a), (b), and (c). Then the manifold $\mathcal{F}$ has finite curvature everywhere.

**Proof.** We first calculate $\rho(x, y, z)$ where $x = F(\alpha)$ and $z = F(\beta)$ with $\alpha, \beta \in \mathbb{R}^n$, and $y$ lies on $\Pi(x)$. We pick the element $y(z) \in \Pi(x)$ where $y(z) = x + t(y - x), t \in \mathbb{R}^1$, so that $\|y(z) - x\| = \|y(z) - z\|$. Hence,
\[
\|x\|^2 - 2(x + t(y - x), x) = \|z\|^2 - 2(x + t(y - x), z).
\]
Solving for $t$, we obtain
\[
t = \frac{\|z\|^2 + \|x\|^2 - 2(x, z)}{2(y - x, z - x)} = \frac{\|z - x\|^2}{2(y - x, z - x)},
\]
and
\[
\rho(x, y, z) = \|y(z) - z\| = \|y(z) - x\| = \frac{\|z - x\|^2\|y - x\|}{2\|(y - x, z - x)\|}.
\]
Fix $x \in \mathcal{F}$. To show that $\mathcal{F}$ has finite curvature at $x$, we will bound $\rho(x, y, z)$ from below. Subtracting and adding $\langle y - x, F'(\alpha) \cdot (\beta - \alpha) \rangle$ in the denominator of $\rho(x, y, z)$ yields
\[
\langle y - x, z - x \rangle = \langle y - x, (z - x) - F'(\alpha) \cdot (\beta - \alpha) \rangle
\]
since \( y \) is in \( \mathbb{N}(x) \). Using the Schwarz inequality, we have

\[
|\langle y - x, z - x \rangle| \leq \|y - x\| \|z - x - F'(\alpha) \cdot (\beta - \alpha)\|.
\]

Hence,

\[
\rho(x, y, z) \geq \frac{\|z - x\|^2}{2\|z - x - F'(\alpha) \cdot (\beta - \alpha)\|}.
\]

Next, we want to find an upper bound for the above denominator when \( z \) is close to \( x \). Since \( F \) is twice continuously Fréchet differentiable in a neighborhood of \( \alpha \), we have (cf. [2, pp. 99, 180])

\[
\|F(\beta) - F(\alpha) - F'(\alpha) \cdot (\beta - \alpha)\|
\leq \left\| \int_0^1 (1 - s)F''(\alpha + s(\beta - \alpha)) \, ds \right\| \cdot (\beta - \alpha)^{(2)} \leq c_1|\beta - \alpha|^2
\]

for some positive constant \( c_1 \) independent of \( \beta \) if \( \beta \) is close to \( \alpha \). Here, \((\beta - \alpha)^{(2)}\) stands for \( ((\beta - \alpha), (\beta - \alpha)) \) and \( |\cdot| \) is a norm in \( \mathbb{R}^n \). This gives, for \( \beta \) close to \( \alpha \) (or \( z \) close to \( x \)),

\[
\rho(x, y, z) \geq \frac{\|F(\beta) - F(\alpha)\|^2}{2c_1|\beta - \alpha|^2}.
\]

It is known (cf. [1, p. 219], [4]) that

\[
F(\beta) - F(\alpha) = \lim_{i \to \infty} \left( \sum_{j=1}^{i} t_{ij} F'(\gamma_{ij}) \right) \cdot (\beta - \alpha)
\]

when \( t_{ij} \geq 0, \sum_{j=1}^{i} t_{ij} = 1 \), and \( \gamma_{ij} \) lies on the line segment formed by \( \alpha \) and \( \beta \), \( 1 \leq j \leq i, \ i = 1, 2, \cdots \). By assumption (c), we see that there is a positive constant \( c_2 \) independent of \( \beta \) so that

\[
\|F'(\alpha) \cdot (\beta - \alpha)\| \geq c_2|\beta - \alpha|
\]

for all \( \beta \). From the continuity of \( F' \), it follows that when \( \beta \) is close to \( \alpha \),

\[
\left\| \lim_{i \to \infty} \sum_{j=1}^{i} t_{ij} F'(\gamma_{ij}) - F'(\alpha) \right\| \leq c_2/2.
\]

Thus, for \( \beta \) close to \( \alpha \),

\[
- \left\| \lim_{i \to \infty} \left( \sum_{j=1}^{i} t_{ij} F'(\gamma_{ij}) \right) \cdot (\beta - \alpha) \right\| + \|F'(\alpha) \cdot (\beta - \alpha)\| \leq \frac{1}{2} c_2|\beta - \alpha|.
\]
or

\[ \frac{1}{2}c_2 |\beta - \alpha| \leq \lim_{i \to \infty} \left\| \sum_{j=1}^{i} t_{ij} F'(y_{ij}) \cdot (\beta - \alpha) \right\| = \| F(\beta) - F(\alpha) \|, \]

so that

\[ \rho(x, y, z) = \frac{c_2 |\beta - \alpha|^2}{2c_1 |\beta - \alpha|^2} = c_3 > 0 \]

for all \( z = F(\beta) \) close to \( x = F(\alpha) \). Hence,

\[ \rho(x, y) = \lim_{z \to x} \inf_{z \in F} \rho(x, y, z) \geq c_3 \]

and \( \rho(x) = \inf_{y \in \Omega(x)} \rho(x, y) \geq c_3 \), or \( \sigma(x) \leq 1/c_3 < \infty \). This completes the proof of the theorem.

3. Unique and nonunique best approximation, and an application. As an immediate consequence of Theorem 1 above and Theorem 4.1 in [4], we have the following:

**Theorem 2.** Let \( F : \mathbb{R}^n \to H \) satisfy (a), (b), and (c). There is a neighborhood \( U \) of \( \Omega \) such that each \( u \in U \) has a unique best approximation from \( F \).

Theorems 1 and 2 could have been stated in terms of “pieces” of manifolds in the following way. Let \( \Omega \) be an open subset of \( \mathbb{R}^n \) and let \( F : \Omega \to H \) satisfy (a), (b), and (c) with the exception that (c) is modified to (c') by changing “for all \( \alpha \in \mathbb{R}^n \)” to “for all \( \alpha \in \Omega \)”. In particular, we have the following

**Corollary.** Let \( \Omega \) be an open subset of \( \mathbb{R}^n \) and let \( F : \Omega \to H \) satisfy (a), (b), and (c'). Then \( F(\Omega) = F \) has a neighborhood \( U \) so that every \( u \in U \) has a unique best approximation in \( F \).

The proof of this corollary requires no new idea and will be omitted. As an example, consider the spline mapping \( F : R^{2N+n+1} \to L^2[0,1] \) defined by

\[ F(\alpha) = F(\alpha_0, \ldots, \alpha_{2N+n}) = \sum_{j=0}^{n} \alpha_j t^j + \sum_{j=1}^{N} \alpha_{n+j}(t - \alpha_{N+n+j})^n \]

where for a real number \( t, (t)_+ = \max(t, 0) \). Let \( \Omega_1 = \{ \alpha \in R^{2N+n+1} : 0 < \alpha_{n+1} \leq \cdots \leq \alpha_{n+2N}, \alpha_j \neq 0 \text{ for } n+1 \leq j \leq n+N \} \) and \( \Omega = \{ \alpha \in \Omega_1 : \).

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Clearly, $F: \Omega \to L^2[0, 1]$ satisfies (a), (b), and (c') if $n \geq 3$. Indeed, by considering $a \in \Omega_1$, we can guarantee that (c') holds, and in order to guarantee that (a) holds, we must bound $a \in \Omega_1$, and hence, we restrict ourselves to $\Omega$. Finally, for $n \geq 3$, it is clear that (b) holds. Thus the corollary applies and we may conclude that $\mathcal{F} = F(\Omega)$ has a neighborhood $\mathcal{U}$ of uniqueness. That is, every $L^2[0, 1]$ function in $\mathcal{U}$ has a unique spline approximation in $L^2[0, 1]$ from $\mathcal{F}$. The reader can see from the method of proof that in this example the constants $c_1$ and $c_2$, which determine $\mathcal{U}$, can be estimated.

These results are in some sense surprising since they are not true in general when the range of $F$ is just a uniformly smooth Banach space. For example, let $l^p(2)$, $1 \leq p < \infty$, be the two-dimensional space $R^2$ with the norm $\| (x, y) \| = (|x|^p + |y|^p)^{1/p}$ and let $F: R^1 \to l^p(2)$ satisfy (a), (b), and (c) with $F(t) = (\cos t, \sin t)$ in a neighborhood $\Omega$ of $t = 0$. Then $F$ is clearly infinitely Fréchet differentiable in $\Omega$. If $p > 2$ and $\epsilon > 0$ is small enough, it is easy to see that the points $(s, 0)$ with $1 - \epsilon < s < 1$ do not have unique best approximations from $\mathcal{F} = F(\Omega)$. Since finite curvature implies the existence of a neighborhood of uniqueness, we can conclude that $\mathcal{F}$ has infinite curvature at $t = 0$.

REFERENCES


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