A TENSOR APPROACH TO INTERPOLATION
PHENOMENA IN DISCRETE ABELIAN GROUPS
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ABSTRACT. We show that a tensor algebra setting is an efficient tool that separates Sidonicity from other interpolation properties.

In this note we show that a tensor algebra setting \cite{11} is a convenient framework where Sidonicity can be efficiently separated from other interpolation properties. Non-Sidon A(\(p\))-sets in any \(\Gamma\) (see A below) were first obtained in \cite{3} and \cite{5}; our proof, similar to the one in \cite{3, Théorème 5, p. 359} is more direct. The Rosenthal property (B) of some non-Sidon sets was first displayed in \cite{9}, where an appeal was made to the notion of "sup-norm partitions" (see also \cite{1} and \cite{2}). C was first observed in \cite{4}.

In what follows below, \(\Gamma\) is a discrete abelian group, and \(\hat{\Gamma} = G\). Without loss of generality, we assume that \(\Gamma\) is a countable group. We refer to \cite{11} for standard notation and facts. Let \(E \subseteq \Gamma\); we set

\[
A(E) = L^1(\Gamma) \setminus \{f \in L^1(\Gamma): \hat{f} = 0 \text{ on } E\},
\]

and

\[
B(E) = M(\Gamma) \setminus \{\mu \in M(\Gamma): \hat{\mu} = 0 \text{ on } E\}.
\]

If \(K(\Gamma)\) is a subspace of \(L^1(\Gamma)\), we set

\[
K_E(\Gamma) = \{f \in K(\Gamma): \hat{f} = 0 \text{ off } E\}.
\]

Definitions. (a) \(E \subseteq \Gamma\) is a Sidon set if \(L^\infty_E(G) = A_E(G) (= l^1(E))\); equivalently, \(A(E) = c_0(E)\).

(b) Let \(1 < p < \infty\). \(E \subseteq \Gamma\) is a \(\Lambda(p)\)-set if \(L^1_E(G) = L^p_E(G)\).

(c) \(E\) is a Rosenthal set if \(L^\infty_E(G) = C_E(G)\).

(d) Let \(1 \leq p < 2\). \(E\) is a \(p\)-Sidon set if \(C_E(G) \subseteq l^p(E)\).

If \(E\) is a Sidon set, then \(E\) is a Rosenthal set, a \(p\)-Sidon set, and a \(\Lambda(p)\)-set, for all \(p\). The first two claims are trivial to verify, whereas the third is not (see 5.7.7 of \cite{11}, and A below). 

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Definition. We say that $S \subset \Gamma$ is a dissociate set if whenever the equality \( \sum_{j=1}^{N} \omega_j \alpha_j = 0 \), where \( \omega_j = -1, 0, 1 \), and \( \{\alpha_j\}_{j=1}^{N} \subset S \), implies that \( \omega_j = 0 \) for all \( j \).

It is well known that every infinite discrete abelian group contains a dissociate set.

Definition. Let $S_1$ and $S_2$ be any countably infinite sets. We set
\[
\ell^\infty(S_1) \otimes \ell^\infty(S_2) = \{ \phi \in \ell^\infty(S_1 \times S_2) : \phi = \sum f_j g_j, \quad f_j \in \ell^\infty(S_1), \quad g_j \in \ell^\infty(S_2), \quad \text{and} \quad \sum \|f_j\| \|g_j\| < \infty \}.
\]
We define $c_0(S_1) \otimes c_0(S_2)$ similarly, and for $\phi \in \ell^\infty(S_1 \times S_2)$ or $c_0(S_1 \times S_2)$ we set $\|\phi\|_\infty = \inf \{ \sum \|f_j\| \|g_j\|_\infty : \phi = \sum f_j g_j \}$. We refer to [12] for a study of the above tensor algebras. We shall need the following:

Fact V (Theorem 3.1 of [11]). Let $E$ and $F$ be infinite disjoint subsets of $\Gamma$ so that $E \cup F$ is dissociate. Then $A(E + F) \approx c_0(E) \otimes c_0(F)$, and $\ell^\infty(E) \otimes \ell^\infty(F)$ is a closed subalgebra of $B(E + F)$. (Note that we can freely associate functions in $\ell^\infty(E + F)$ with functions in $\ell^\infty(E \times F)$.)

By the above, since $c_0(E) \otimes c_0(F) \subseteq c_0(E + F)$ (see, for example, Lemma VIII.10.5 of [6]), $E + F$ is not a Sidon set.

In what follows below, $E = \{\lambda_j\}_{j=1}^{\infty}$ and $F = \{\nu_j\}_{j=1}^{\infty}$ are infinite disjoint subsets of $\Gamma$, and $E \cup F$ is a dissociate set.

A. $E + F (E + E)$ is a $\Lambda(p)$-set for all $p$. The proof that every Sidon set in $\Gamma$ is a $\Lambda(p)$-set for all $p$ (see 5.7.7 in [11]) is based on the prior knowledge that an infinite independent set in $\bigoplus \mathbb{Z}^2$ is a $\Lambda(p)$-set for all $p$. We follow a similar route by making use of

Lemma. There exists $\Gamma_1$, an infinite discrete abelian group ($\Gamma_1 = H$), and $S_1 = \{\alpha_j\}_{j=1}^{\infty}$, $S_2 = \{\beta_j\}_{j=1}^{\infty}$, two infinite disjoint subsets of $\Gamma_1$, so that $S_1 \cup S_2$ is dissociate and $S_1 + S_2$ is a $\Lambda(p)$-set for all $p$.

See [9], for example.

Now, let $p > 2$ and let $f$ be any trigonometric polynomial in $L^1_{E + F}(G)$:
\[
f(g) = \sum a_{ij}(\lambda_i + \nu_j, g).
\]
We use $S_1$ and $S_2$ in $\Gamma_1$ of the above Lemma to perturb $\hat{f}$:
\[
f_h(g) = \sum a_{ij}(\alpha_i, h)(\beta_j, h)(\lambda_i + \nu_j, g) \quad \text{for all} \quad h \in H.
\]
We now note that for each $h \in H$, there exists $\mu = \mu_h$ in $M(G)$ so that
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\begin{align*}
\mu_h(\lambda_j + \nu_j) &= (a_{ij}, h)(\beta_j, h), \\
\text{and } \sup_{p} \|\mu_p\|_M &= b < \infty. \text{ This follows immediately from Fact V (see Remark (i)).}
\end{align*}

We proceed exactly as in 5.7.7 of [11]: \( f = f_h \ast \mu_h \), and therefore

\[ \|f\|_p \leq \|f_h\|_p \|\mu_h\|_M \leq b \|f_h\|_p. \]

That is,

\[ \int_G |f(g)|^p \, dg \leq b^p \int_G \left| \sum_{i=1}^n a_{ij}(\lambda_i, h)(\beta_j, h)(\lambda_j + \nu_j, g) \right|^p \, dg. \]

We integrate both sides over \( H \), interchange the order of integration, and apply the above Lemma. \( \square \)

**Remarks.**

(i) We do not have to appeal to Fact V: We can obtain (*) directly by considering the Riesz product whose transform equals \( (\alpha_i, h) \) at \( \lambda_i \) and \( (\beta_j, h) \) at \( \nu_j \).

(ii) The \( \Lambda(p) \) constants of \( E + F \) are inherited from \( S_1 + S_2 \) in \( \Gamma_1 \). In fact, by interchanging the roles of \( G \) and \( H \) in the above proof, we see that the behavior of the \( \Lambda(p) \) constants of \( E + F \) is the same in all groups.

(iii) The above argument can be easily modified to prove that \( E + E \) is a \( \Lambda(p) \)-set for all \( p \).

B. \( E + F \) is a Rosenthal set (see also [8, p. 251]). We note that

\[ (c_0(E) \otimes c_0(F))^* = \{ (a_{mn}) : \exists C > 0 \text{ so that } \left| \sum a_{mn} x_{mn} y_n \right| \leq C \|x\|_\infty \|y\|_\infty \text{ for all } x, y \in c_0(E), c_0(F) \text{ respectively} \}. \]

But, \( (c_0(E) \otimes c_0(F))^* = \mathcal{L}_{E+F}(G)^\wedge \), and since the right-hand side of (1) can be viewed as a sup-norm closure of trigonometric polynomials with spectrum in \( E + F \), our assertion follows. \( \square \)

C. \( E + F \) is a 4/3-Sidon set. By (1) above, the assertion is precisely the content of the third inequality in Theorem 1 (1) of Littlewood [7]:

\[ \left( \sum \sum |a_{mn}|^{4/3} \right)^{3/4} \leq A \left( \sup_{x,y : \|x\|_\infty, \|y\|_\infty = 1} \left| \sum a_{mn} x_m y_n \right| \right). \]

Littlewood observes in [7] that the 4/3 is sharp.

**REFERENCES**


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