AN INVARIANT SUBSPACE THEOREM

JOHN DAUGHTRY

ABSTRACT. If $AY - YA$ has rank one for some compact $Y$, then $A$ has a nontrivial invariant subspace.

Let $X$ be a complex Banach space and $B(X)$ the set of bounded linear operators on $X$. Lomonosov has proved that if $A$ and $Y$ belong to $B(X)$ with $Y$ compact and $AY - YA = 0$, then $A$ has an invariant subspace [1]. We have obtained an extension of this result.

Theorem. If $AY - YA$ has rank one for some compact $Y$, then $A$ has a nontrivial invariant subspace.

This result contrasts with the fact that if $A$ does not commute with a trace class operator then $\{AY - YA | Y \text{ is compact}\}$ is uniformly dense in the compacts [4].

The proof of the theorem requires a lemma which may be attributed to David Luenberger although it is incorrectly stated in his paper [2]. The author is grateful to J. P. Williams who pointed out the correct version of the lemma and counterexamples to Luenberger’s stronger version and partial converse.

Lemma (Luenberger). Suppose

\[
TA - BT = C
\]

has rank one for $T$, $A$, and $B$ in $B(X)$. If the largest $A$-invariant subspace of $X$ contained in the kernel of $C$ is $\{0\}$, and the smallest $B$-invariant subspace of $X$ containing the range of $C$ is $X$, then either $T$ is one-to-one or $T$ has dense range.

Proof. Assume that $T$ has nontrivial kernel and nondense range in order to contradict the hypothesis.

Choose $x \neq 0$ in the kernel of $T$. Then $TAx = Cx$, which implies either $Cx \neq 0$ or $Ax \in \ker T$. The second alternative together with (1) yield

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$TA^2x = CAx$, hence $CAx \neq 0$ or $A^2x \in \ker T$. We may repeat this argument indefinitely. Because the span of \{A^n x\} cannot be an invariant subspace for $A$ contained in the kernel of $C$, there exists $y \in \ker T$ with $Cy \neq 0$.

Since $Cy = TAx$ we conclude that the kernel of $A^*T^*$ annihilates the (one-dimensional) range of $C$, or $\ker A^*T^* \subseteq \ker C^*$. Then $A^*T^* - T^*B^* = C^*$ implies that $\ker A^*T^* \subseteq \ker T^*B^*$. For $x^* \in \ker A^*T^*$ we have $(A^*T^*)B^*x^* = A^*(T^*B^*)x^* = 0$, so $A^*T^*$ is a $B^*$-invariant subspace contained in the nullspace of $C^*$. It follows that the closure of the range of $TA$ is a $B$-invariant subspace containing the range of $C$, completing the proof of the lemma.

Begin the proof of the theorem by assuming that $A$ has no invariant subspace, $Y$ is compact, and $C = Ay - Ya$ has rank one. By one version of Lomonosov's theorem [3], there exists an operator $B$ commuting with $A$ such that $BYg = g$ for some nonzero $g$ in $X$. Then

$$BC = B(Ay - Ya) = A(By) - (By)A = A(By - l) - (By - l)A$$

has rank one ($B$ has trivial kernel since it commutes with $A$). Yet $BY - l$ has nontrivial kernel and by the Fredholm alternative it has nondense range. This conclusion is contrary to the lemma.

Note added in proof. Perhaps it should be remarked that if $Y$ has rank one then the rank of $AY - YA$ is no greater than two. Thus the task of replacing "rank one" by "rank two" in the hypothesis is equivalent to solving the invariant subspace problem.

REFERENCES


DEPARTMENT OF MATHEMATICS, SWEET BRIAR COLLEGE, SWEET BRIAR, VIRGINIA 24595