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THE HUGHES CONJECTURE AND GROUPS WITH ABSOLUTELY REGULAR SUBGROUPS OR ECF-SUBGROUPS

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ABSTRACT. Let G be a finite p-group and $H_p(G)$ the subgroup generated by the elements of G of order different from p. Hughes conjectured that if $G > H_p(G) > 1$, then $|G: H_p(G)| = p$. Although the conjecture is not true for all G, it is shown here that if G has a subgroup L such that $|L: L^p| \leq p^{p-r}$ $(r \geq 1)$ and $p^r \leq |G: L| \leq p^{r+p}$ or an ECF-subgroup L with $|G: L_2| \leq p^{p+2}$, then G satisfies the Hughes conjecture.

Let G be a group, p a prime and $H_p(G)$ the subgroup of G generated by the elements of order different from p. Hughes conjectured that if $G > H_p(G) > 1$, then $|G: H_p(G)| = p$. The conjecture is known to be true for any G with $p = 2$ [6] or $p = 3$ [11] and for any p with G finite and not a p-group [7]. Although there is a finite 5-group with $|G: H_5(G)| = 25$ [12], many authors have given sufficient conditions for the conjecture to hold. We refer the reader to [4] and [5] for summaries of these results.

In an earlier paper [5], the author proved that if G contains a subgroup L such that $|L: L^p| \leq p^{p-r}$ $(r \geq 1)$ and L has index p^r or p^{r+1} in G or a normal ECF-subgroup L with $|G: L_2| \leq p^{p+2}$, then G satisfies the Hughes conjecture. In this paper these two results and Theorem 3 of [4] are generalized.

For the remainder of the paper G denotes a finite p-group. If G has class $c = c(G)$, we use $G = G_1 > G_2 > \cdots > G_{c+1} = 1$ to denote the lower central series of G. Let $G^p = \langle g^p | g \in G\rangle$. Following Blackburn, we say G is absolutely regular if $|G: G^p| \leq p^{p-1}$ and we call G an ECF-group if $|G_i/G_{i+1}| = p$ for $i = 2, \ldots, c$ and $\exp(G/G_2) = p$. An ECF-group G with $|G: G_2| = p^2$ is called a group of maximal class.

Theorem 1. *Suppose G has a subgroup L such that $|L: L^p| \leq p^{p-r}$ $(r \geq 1)$ and $p^r \leq |G: L| \leq p^{r+p}$. Then G satisfies the Hughes conjecture.*

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Proof. Suppose G is a minimal counterexample. By the result mentioned in the second paragraph we have $p^{r+2} \le |G: L| \le p^{r+p}$. Let N be a central subgroup of G of order p. Then $N \le H_p(G)$ so that $H_p(G/N) \le H_p(G)/N$ and $|G/N: H_p(G/N)| \ge |G: H_p(G)| > p$. Also $p^r \le |G/N: LN/N| \le p^{r+p}$ and $|LN/N: (LN/N)^p| \le |L: L^p| \le p^{p-r}$. Thus G/N satisfies the hypothesis of the theorem and it follows that $H_p(G/N) = 1$. Hence $|L^p| \le p$, so that $|L| \le p^{p-r+1}$ and $|G| \le p^{2p+1}$. But Corollary 3 in [9], the theorem in [10] and Theorem 1 in [4] together show that a counterexample must have order at least p^{2p+2}. This contradiction proves the theorem.

Theorem 2. *Suppose G has an ECF-subgroup L such that $|G: L_2| \le p^{p+2}$. Then G satisfies the Hughes conjecture.*

Proof. Suppose G is a counterexample. As mentioned in the proof of Theorem 1 we must have $|G| \ge p^{2p+2}$. It follows that $c(L) > p$. There exists a subgroup M of L such that $M_i = L_i$ for all $i \ge 2$ and $|M: M_2| = p^2$ [2, p. 65]. Furthermore, M has a subgroup N of index p such that $N^p = M_p$ [2, p. 69]. It follows that $|N: N^p| = p^{p-1}$ and $p \le |G: N| \le p^{p+1}$. This contradicts Theorem 1.

A special case of Theorem 2 is worth stating.

Corollary 1. *If G has a subgroup L of index at most p^p which is a group of maximal class, then G satisfies the Hughes conjecture.*

Corollary 2. *Suppose G possesses a subgroup L of index at most p^p and there is an integer s satisfying $p^p \le p^s \le |L|$ such that all the normal subgroups of L of order p^s are absolutely regular. Then G satisfies the Hughes conjecture.*

Proof. A theorem of Blackburn [3, p. 8] shows that either L is absolutely regular or L is a group of maximal class. By Corollary 1, we may assume that $|L: L^p| \le p^{p-1}$. If $G = L$, then G is regular [8, p. 332] and G satisfies the Hughes conjecture (see [5]). If $L < G$, then G satisfies the hypothesis of Theorem 1 with $r = 1$.

A well-known theorem of P. Hall [8, p. 334] says that if G is not regular then G has a normal subgroup of order p^{p-1} and exponent p. For groups which do not satisfy the Hughes conjecture much more can be said.

Corollary 3. *If G does not satisfy the Hughes conjecture, then for every subgroup L of G of index p^p the number of subgroups of L of order*

p^p and exponent p is congruent to 1 modulo p. Furthermore, every normal subgroup of index p^p in G contains a subgroup of order p^p and exponent p which is normal in G.

Proof. A pair of results of Berkovič [1, pp. 826, 828] shows that the existence of a subgroup L which does not have the required properties implies that L is either absolutely regular or a group of maximal class. Thus, by either Theorem 1 or Theorem 2, we have a contradiction.

Theorem 3 of [4] says that if $|G| = p^n$ and for some fixed r with $3 \le r < n$ all the normal subgroups of G of order p^r are generated by two elements, then G satisfies the Hughes conjecture. That theorem was generalized by Theorem 6 of [5] and we conclude this paper with a different generalization.

Theorem 3. *Suppose G has a subgroup L of index at most p^{p+2} in G and every normal subgroup of L of order p^r has two generators where r is fixed and $p^3 \le p^r < |L|$. Then G satisfies the Hughes conjecture.*

Proof. Suppose G is a counterexample. Then, since $|G| \ge p^{2p+2}$ (see proof of Theorem 1) and $|G: L| \le p^{p+2}$ we have $|L| \ge p^p$. Also Theorem 6 of [5] shows that $|G: L| \ge p^2$. First suppose $p^r = |L|/p$ and $|L| \ne 5^5$. Since $p \ge 5$, a theorem of Blackburn [3, p. 19] shows that L is metacyclic or $|L: L^p| = p^3$. If L is metacyclic, then $|L: L^p| \le p^2$ [8, p. 337] so that in either case we have $|L: L^p| \le p^{p-2}$ and Theorem 1 shows that we have derived a contradiction. Now suppose $p^r = |L|/p$ and $|L| = 5^5$. If $|L^5| = 1$ or 5, there is a normal subgroup N of L of order 5 such that L/N has exponent 5. Since every group of order 5^4 has an Abelian subgroup of order 5^3, let M/N be a subgroup of L/N with this property. Then $|M| = 5^4$ so that M/N is generated by two elements and is elementary Abelian. This is clearly impossible. Thus $|L: L^5| \le 5^3$ and Theorem 1 again yields a contradiction.

Finally, suppose $p^r < |L|/p$. Then another theorem of Blackburn [3, p. 16] implies that L is metacyclic or the elements of L of order at most p form a subgroup E of order p^3 and L/E is cyclic. If the latter is true, then $c(L) \le 4 < p$ so L is regular. Then $|L: L^p| = |E| = p^3 \le p^{p-2}$ [8, p. 327]. Thus, in either case, we have $|L: L^p| \le p^{p-2}$ in contradiction to Theorem 1.

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