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A REAL PLACE ON THE REAL NUMBER FIELD IS TRIVIAL

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ABSTRACT. This paper proves that any real place from the real number system must be trivial. The result extends to a real place on any archimedean real closed field.

A *place* ϕ from field F to field K is a ring homomorphism from a valuation subring V of F into K whose kernel is the unique maximal ideal of V. A *real field* is an ordered field in which -1 is not a sum of squares. A *real place* is a place whose codomain is a real field. A real field is *quadratically closed* if its positive elements are squares; it is *archimedean* in case for every element x there exist rational numbers r and s such that $r < x < s$.

Let Q and R denote the fields of rational numbers and real numbers, respectively.

Theorem. *Let F be an archimedean quadratically closed real field, and let K be any real field. If ϕ is a place from F to K, then ϕ is an isomorphism.*

Proof. We may consider F a subfield of R [1]. Let V be the domain of ϕ. Since $\mathrm{char}(K) = 0$, $Q \cap V$ is a valuation subring of Q whose residue field has characteristic zero. Hence $Q \cap V = Q$, and ϕ is trivial on Q.

Suppose there exists $a \in F$, $a \neq 0$, such that $\phi(a) = 0$. We may suppose $a > 0$. Then there exists $b \in Q$, $b \neq 0$, such that $b^2 < a$. Then $a - b^2 > 0$, so there exists $c \in F$ such that $a - b^2 = c^2$. Since $a, b \in V$, we have $c^2 \in V$; if $c \notin V$, then $c^{-1} \in V$ and $c = c^{-1} \cdot c^2 \in V$; hence $c \in V$. Then $-\phi(b)^2 = \phi(a - b^2) = \phi(c)^2$, or $-1 = \phi(c/b)^2$, contradicting the fact that K is real. Hence $\ker \phi = (0)$, and ϕ is trivial on F.

Corollary. *A real place on R is trivial.*

REFERENCE

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