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POLYNOMIAL RINGS OVER A COMMUTATIVE VON NEUMANN REGULAR RING

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ABSTRACT. It is shown that the annihilator of each finitely generated ideal of $R[\{X_\lambda\}_{\lambda\in\Lambda}]$, where R is a commutative von Neumann regular ring with identity, is principal; this generalizes a recent result of P. J. McCarthy.

In a recent paper [5], P. J. McCarthy showed that the ring of polynomials in one indeterminate over a commutative von Neumann regular ring R with identity is semihereditary.[1] Based upon results of W. Vasconcelos [8] and C. U. Jensen [3], McCarthy's proof rested upon the establishment of the following result, which we label as Theorem A.

Theorem A. *If $f(X) = a_0 + a_1X + \cdots + a_nX^n$ is in $R[X]$ and if e_i is an idempotent generator of the ideal a_iR for each i, then the annihilator of $f(X)$ is the principal ideal of $R[X]$ generated by $(1-e_0)(1-e_1)\cdots(1-e_n)$.*

In this note we prove a result (Theorem C) that includes Theorem A; we begin with more general considerations. Let S be a commutative ring, let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a set of indeterminates over S, and let f be an element of the polynomial ring $S[\{X_\lambda\}]$. If A_f is the ideal of S generated by the coefficients of f, then it is clear that each element of $B[\{X_\lambda\}]$, where B is the annihilator of the ideal A_f, annihilates f. In the next result we give sufficient conditions in order that $B[\{X_\lambda\}]$ should be the annihilator of f.

Proposition B.[2] *In order that $B[\{X_\lambda\}]$ should be the annihilator of f, it is sufficient that either of the following conditions is satisfied.*

(1) *The ideal A_f is idempotent.*

(2) *The ring S contains no nonzero nilpotent element.*

Proof. Assume that (1) is satisfied. Since A_f is finitely generated, it

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[1] A statement of this result is also contained in [4].

[2] A form of this result, for power series rings, is contained in [1]; the paper [7], by J. Ohm and D. Rush, considers problems related to Proposition B.

is principal and is generated by an idempotent element; moreover, R is the direct sum of the ideals A_f and B and $R[\{X_\lambda\}] = A_f[\{X_\lambda\}] \oplus B[\{X_\lambda\}]$. Since f is in $A_f[\{X_\lambda\}]$ and since the coefficients of f generate A_f as an ideal of the ring A_f with identity, it follows from McCoy's theorem [6] that f is not a zero divisor of the ring $A_f[\{X_\lambda\}]$. Consequently, $B[\{X_\lambda\}]$ is the annihilator of f, as asserted.

If S contains no nonzero nilpotent element, then (0) is the intersection of a family $\{P_\alpha\}$ of proper prime ideals of S; for each α we denote by ϕ_α the canonical homomorphism of $S[\{X_\lambda\}]$ onto $(S/P_\alpha)[\{X_\lambda\}]$. If the polynomial g annihilates f, then for each α, $\phi_\alpha(f)\phi_\alpha(g) = 0$ in $(S/P_\alpha)[\{X_\lambda\}]$, an integral domain. Hence $\phi_\alpha(f) = 0$ or $\phi_\alpha(g) = 0$, and in any case, $g_i A_f \subseteq \bigcap_\alpha P_\alpha = (0)$ and $g_i \in B$, as we wished to prove.

If a and b are idempotent elements of a commutative ring S, then $Sa \cap Sb = Sab$; moreover, if U is an ideal of S, then the annihilator of U is the intersection of the family of annihilators of elements of a generating set of U. These two elementary observations, together with Proposition B, yield Theorem C.

Theorem C. *Let S be a commutative von Neumann regular ring with identity, let $f_1, f_2, \cdots, f_m$ be elements of the polynomial ring $S[\{X_\lambda\}]$, let $\{a_i\}_1^k$ be the set of coefficients of the polynomials $f_1, f_2, \cdots, f_m$, and for each i between 1 and k, let e_i be an idempotent generator of the ideal Sa_i of S. The annihilator of the ideal*[3] *of $S[\{X_\lambda\}]$ generated by $\{f_j\}_{j=1}^m$ can be described as the principal ideal of $S[\{X_\lambda\}]$ generated by $(1 - e_1)(1 - e_2) \cdots (1 - e_k)$ or as the principal ideal of $S[\{X_\lambda\}]$ generated by $1 - e$, where e is an idempotent generator of the ideal $(a_1, \cdots, a_k)$ of S.*

We remark that Proposition B and Theorem C are true, more generally, if the polynomial ring is replaced by the semigroup ring of an arbitrary torsion-free cancellative abelian semigroup with zero. Moreover, an even shorter proof of Proposition B can be obtained by the application of the Dedekind-Mertens lemma, which is known to generalize to such semigroup rings.

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[3]If $|\{X_\lambda\}| = 1$, then each finitely generated ideal of $S[\{X_\lambda\}]$ is principal [2], but this is not the case, of course, for polynomial rings in more than one variable.

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