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INVARIANT MEANS ON ALMOST PERIODIC FUNCTIONS AND EQUICONTINUOUS ACTIONS

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ABSTRACT. Let S be a topological semigroup such that the almost periodic functions on S have a left invariant mean (this is the case, for example, when S has finite intersection property for closed right ideals). Then whenever S acts equicontinuously on a compact Hausdorff space X, there exists a compact group G of homeomorphisms acting equicontinuously on a retract Y of X such that S has a common fixed point in X if and only if G has a common fixed point in Y. This result generalises some recent work of T. Mitchell. As an application, we show that whenever S acts equicontinuously on the closed unit interval I, then I contains a common fixed point for S.

1. **Introduction.** Let S be a semigroup of equicontinuous self maps of X, a compact Hausdorff space. Recently, T. Mitchell [6] showed that if S has finite intersection property for right ideals, then there is a compact group G of homeomorphisms of a retract Y of X with the property that S has a common fixed point in X if and only if G has a common fixed point in Y. It is the purpose of this note to modify Mitchell's result. In particular, we show that the same conclusion also holds when S satisfies an analytic condition (which is implied by Mitchell's algebraic condition), namely, the existence of a left invariant mean on the space of almost periodical functions on S.

2. **Some notations.** For the rest of this paper S will be a fixed topological semigroup (with separately continuous multiplication).

Let X be a compact Hausdorff space and let $\mathfrak{U}$ be the unique uniformity on X. Then S is said to *act equicontinuously* on X if there exists a continuous mapping from the product space $S \times X \to X$, denoted by (s, x)

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$\to s \cdot x$, such that

(1) $s \cdot (t \cdot x) = (st) \cdot x$ for all $s, t \in S$;

(2) for each $x \in S$, $U \in \mathfrak{U}$, and $y \in X$, there exists $V \in \mathfrak{U}$ such that $(x, y) \in V$ implies $(s \cdot x, s \cdot y) \in U$ for all $s \in S$.

Let $C(X, X)$ denote the space of continuous functions from X to X with the topology of uniform convergence on X, and let $\sigma: S \to C(X, X)$ be given by $(\sigma s)(x) = s \cdot x$ for all s in S and x in X. Then $\overline{S}$, the closure of S in $C(X, X)$ [3, p. 270] is a compact topological semigroup with jointly continuous multiplication.

For any topological space Y, $C(Y)$ will denote the space of bounded real-valued functions on Y. A function f in $C(S)$ is *almost periodic* if $\{l_a f;\ a \in S\}$ is relatively compact in the sup norm topology of $C(S)$, where $(l_a f)(s) = f(as)$ for all $a, s \in S$. Then, as known, $AP(S)$ the space of almost periodic functions on S is translation invariant, sup norm closed and containing constants (see [2, p. 80]). An element ϕ in $AP(S)^*$, the continuous dual of $AP(S)$, is a *mean* if $\|\phi\| = 1$ and $\phi(f) \geq 0$ whenever $f \in AP(S)$ and $f \geq 0$. If, in addition, $\phi(l_a f) = \phi(f)$ for all $a \in S$ and $f \in AP(S)$, then ϕ is a LIM (*left invariant mean*).

Let $\Delta(S)$ denote the set of all means ϕ of $AP(S)$ which are *multiplicative*, i.e. $\phi(fg) = \phi(f)\phi(g)$ for all $f, g \in AP(S)$. Then, as known, $\Delta(S)$ is weak*-compact and the set $\{p_a;\ a \in S\}$, where $p_a(f) = f(a)$ for all $f \in AP(S)$, of point measures on $AP(S)$ is weak*-dense in $\Delta(S)$. Furthermore, the Aren's product, $(\rho \odot \mu)(f) = \rho(h)$ where $h(s) = \mu(l_s f)$ for all $s \in S$, $\rho, \mu \in \Delta(S)$ renders $\Delta(S)$ into a compact topological semigroup with the weak* topology and the multiplication in $\Delta(S)$ is even jointly continuous (see Pym [7, §5]).

A topological semigroup S is *left reversible* if any two *closed* right ideals in S have nonempty intersection. As known, if S is left reversible, then $AP(S)$ has a LIM. However, the class of topological semigroups S for which $AP(S)$ has a LIM and yet S is not left reversible is huge (see for example [5, Remark 3.4]).

3. **The main theorem.** The proof of our main theorem is based on the following two lemmas. The first one is known.

Lemma 1 [5, Lemma 3.1]. *If S acts equicontinuously on a compact Hausdorff space X, and $x \in X$, then $T_x f \in AP(S)$ for any f in $C(X)$, where $(T_x f)(s) = f(s \cdot x)$ for all $s \in S$.*

Lemma 2. *If S acts equicontinuously on a compact Hausdorff space X, then $\overline{S}$ is a continuous homomorphic image of the compact semigroup $\Delta(S)$.*

Proof. For each x in X, ϕ in $\Delta(S)$, let $(h\phi)(x)$ be a cluster point of the net $\{s_\alpha \cdot x\}$ in X where $\{p_{s_\alpha}\}$ is a net of point measures on $AP(S)$ conveging to ϕ in the weak* topology. Let $\{p_{t_\beta}\}$ be another net of point measures on $AP(S)$ converging to ϕ in the weak* topology. By passing to subnets if necessary, we may assume that $s_\alpha \cdot x \longrightarrow y$ and $t_\beta \cdot x \longrightarrow z$. If $f \in C(X)$, then it follows from Lemma 1 that $T_x f \in AP(S)$; hence

$$f(y) = \lim_\alpha f(s_\alpha x) = \lim_\alpha p_{s_\alpha}(T_x f) = \phi(T_x f) = \lim_\beta f(t_\beta x) = f(z).$$

Therefore $h : \phi \longrightarrow h\phi$ defines a mapping from $\Delta(S)$ into $\overline{S}$. It is a routine matter to verify that h is a continuous homomorphism of $\Delta(S)$ onto $\overline{S}$.

A topological semigroup S is said to have property (K) if whenever S acts equicontinuously on a compact Hausdorff space X, there exists a compact subgroup G of $\overline{S}$ and a retract Y of X satisfying the following two conditions:

(a) the restriction of G to Y is a group of homeomorphisms from Y onto Y;

(b) G has a common fixed point in Y if and only if S has a common fixed point in X.

Theorem. *If $AP(S)$ has a LIM, then S has property* (K).

Proof. If $AP(S)$ has a LIM, then $\Delta(S)$ is left reversible (see [2, §§5, 6]). Hence by Lemma 2, $\overline{S}$ is also left reversible. Our result now follows from [6, Theorem 1].

Corollary 1. *If $AP(S)$ has a LIM, then whenever S acts equicontinuously on the closed unit interval I, I has a common fixed point for S.*

Proof. Let G be a compact subgroup of $\overline{S}$ and Y a retract of I with properties as stated in (K). Since Y is a closed subinterval of I, it follows from [6, Theorem 2] that G has a common fixed point in Y. Hence S has a common fixed point in I.

Corollary 2. *If S is left reversible, then S has property* (K).

Proof. If S is left reversible, then $AP(S)$ has LIM (see proof of Corollary 3.3 in [5]).

Our main theorem and the two corollaries are due to Mitchell [6] for the case when S is a discrete left reversible semigroup.

An open problem. Does property (K) imply $AP(S)$ has a LIM?

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