INVARIANT MEANS ON ALMOST PERIODIC FUNCTIONS
AND EQUICONTINUOUS ACTIONS

ANTHONY TO-MING LAU

ABSTRACT. Let $S$ be a topological semigroup such that the almost periodic functions on $S$ have a left invariant mean (this is the case, for example, when $S$ has finite intersection property for closed right ideals). Then whenever $S$ acts equicontinuously on a compact Hausdorff space $X$, there exists a compact group $G$ of homeomorphisms acting equicontinuously on a retract $Y$ of $X$ such that $S$ has a common fixed point in $X$ if and only if $G$ has a common fixed point in $Y$. This result generalises some recent work of T. Mitchell. As an application, we show that whenever $S$ acts equicontinuously on the closed unit interval $I$, then $I$ contains a common fixed point for $S$.

1. Introduction. Let $S$ be a semigroup of equicontinuous self maps of $X$, a compact Hausdorff space. Recently, T. Mitchell [6] showed that if $S$ has finite intersection property for right ideals, then there is a compact group $G$ of homeomorphisms of a retract $Y$ of $X$ with the property that $S$ has a common fixed point in $X$ if and only if $G$ has a common fixed point in $Y$. It is the purpose of this note to modify Mitchell's result. In particular, we show that the same conclusion also holds when $S$ satisfies an analytic condition (which is implied by Mitchell's algebraic condition), namely, the existence of a left invariant mean on the space of almost periodical functions on $S$.

2. Some notations. For the rest of this paper $S$ will be a fixed topological semigroup (with separately continuous multiplication).

Let $X$ be a compact Hausdorff space and let $U$ be the unique uniformity on $X$. Then $S$ is said to act equicontinuously on $X$ if there exists a continuous mapping from the product space $S \times X \to X$, denoted by $(s, x)$.
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→ s · x, such that

(1) s · (t · x) = (st) · x for all s, t ∈ S;

(2) for each x ∈ S, U ∈ ℰ, and y ∈ X, there exists V ∈ ℰ such that

(x, y) ∈ V implies (s · x, s · y) ∈ U for all s ∈ S.

Let C(X, X) denote the space of continuous functions from X to X with
the topology of uniform convergence on X, and let σ : S → C(X, X) be given
by (as)(x) = s · x for all s ∈ S and x in X. Then $\widetilde{S}$, the closure of S in
C(X, X) [3, p. 270] is a compact topological semigroup with jointly contin-
uous multiplication.

For any topological space Y, $C(Y)$ will denote the space of bounded
real-valued functions on Y. A function $f$ in $C(S)$ is almost periodic if \{l_{af}; a ∈ S\} is relatively compact in the sup norm topology of $C(S)$, where $(l_{af})(s) = f(as)$ for all $a, s ∈ S$. Then, as known, $AP(S)$ the space of almost periodic functions on S is translation invariant, sup norm closed and containing constants (see [2, p. 80]). An element $\phi$ in $AP(S)^*$, the continuous dual of $AP(S)$, is a mean if $\|\phi\| = 1$ and $\phi(f) ≥ 0$ whenever $f ∈ AP(S)$ and $f ≥ 0$. If, in addition, $\phi(l_{af}) = \phi(f)$ for all $a ∈ S$ and $f ∈ AP(S)$, then $\phi$ is a LIM
(left invariant mean).

Let $Δ(S)$ denote the set of all means $\phi$ of $AP(S)$ which are multiplica-
tive, i.e. $\phi(fg) = \phi(f)\phi(g)$ for all $f, g ∈ AP(S)$. Then, as known, $Δ(S)$ is
weak*-compact and the set \{p_a; a ∈ S\}, where $p_a(f) = f(a)$ for all $f ∈ AP(S)$,
of point measures on $AP(S)$ is weak*-dense in $Δ(S)$. Furthermore, the Aren’s
product, $(ρ ∪ μ)f = ρ(h) \mu(h)$ where $h(s) = μ(l_s f)$ for all $s ∈ S$, $ρ, μ ∈ Δ(S)$ ren-
ders $Δ(S)$ into a compact topological semigroup with the weak* topology and
the multiplication in $Δ(S)$ is even jointly continuous (see Pym [7, §5]).

A topological semigroup $S$ is left reversible if any two closed right
ideals in $S$ have nonempty intersection. As known, if $S$ is left reversible, then
$AP(S)$ has a LIM. However, the class of topological semigroups $S$ for which
$AP(S)$ has a LIM and yet $S$ is not left reversible is huge (see for example
[5, Remark 3.4]).

3. The main theorem. The proof of our main theorem is based on the
following two lemmas. The first one is known.

Lemma 1 [5, Lemma 3.1]. If $S$ acts equicontinuously on a compact Haus-
dorff space $X$, and $x ∈ X$, then $T_x f ∈ AP(S)$ for any $f$ in $C(X)$, where
$(T_x f)(s) = f(s · x)$ for all $s ∈ S$.

Lemma 2. If $S$ acts equicontinuously on a compact Hausdorff space $X$, then $\widetilde{S}$ is a continuous homomorphic image of the compact semigroup $Δ(S)$.

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Proof. For each $x$ in $X$, $\phi$ in $\Delta(S)$, let $(h\phi)(x)$ be a cluster point of the net $\{s_{a^*}x\}$ in $X$ where $\{p_{\alpha}\}$ is a net of point measures on $AP(S)$ converging to $\phi$ in the weak* topology. Let $\{p_{t^*}\}$ be another net of point measures on $AP(S)$ converging to $\phi$ in the weak* topology. By passing to subnets if necessary, we may assume that $s_{a^*}x \to y$ and $t_{b^*}x \to z$. If $f \in C(X)$, then it follows from Lemma 1 that $T_x f \in AP(S)$; hence

$$f(y) = \lim_{a} f(s_{a^*}x) = \lim_{a} p_{s^*}(T_x f) = \phi(T_x f) = \lim_{b} f(t_{b^*}x) = f(z).$$

Therefore $h: \phi \to h\phi$ defines a mapping from $\Delta(S)$ into $\bar{S}$. It is a routine matter to verify that $h$ is a continuous homomorphism of $\Delta(S)$ onto $\bar{S}$.

A topological semigroup $S$ is said to have property (K) if whenever $S$ acts equicontinuously on a compact Hausdorff space $X$, there exists a compact subgroup $G$ of $S$ and a retract $Y$ of $X$ satisfying the following two conditions:

(a) the restriction of $G$ to $Y$ is a group of homeomorphisms from $Y$ onto $Y$;
(b) $G$ has a common fixed point in $Y$ if and only if $S$ has a common fixed point in $X$.

Theorem. If $AP(S)$ has a LIM, then $S$ has property (K).

Proof. If $AP(S)$ has a LIM, then $\Delta(S)$ is left reversible (see [2, §§5, 6]). Hence by Lemma 2, $\bar{S}$ is also left reversible. Our result now follows from [6, Theorem 1].

Corollary 1. If $AP(S)$ has a LIM, then whenever $S$ acts equicontinuously on the closed unit interval $I$, $I$ has a common fixed point for $S$.

Proof. Let $G$ be a compact subgroup of $\bar{S}$ and $Y$ a retract of $I$ with properties as stated in (K). Since $Y$ is a closed subinterval of $I$, it follows from [6, Theorem 2] that $G$ has a common fixed point in $Y$. Hence $S$ has a common fixed point in $I$.

Corollary 2. If $S$ is left reversible, then $S$ has property (K).

Proof. If $S$ is left reversible, then $AP(S)$ has LIM (see proof of Corollary 3.3 in [5]).

Our main theorem and the two corollaries are due to Mitchell [6] for the case when $S$ is a discrete left reversible semigroup.

An open problem. Does property (K) imply $AP(S)$ has a LIM?

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REFERENCES


5. A. Lau, Invariant means on almost periodic functions and fixed point properties, Rocky Mountain J. Math. 3 (1973), 69-76.


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON 7, ALBERTA, CANADA