A LOCAL CHARACTERIZATION OF DARBOUX \( \mathfrak{B} \) FUNCTIONS

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ABSTRACT. A. M. Bruckner and J. B. Bruckner gave the definition of Darboux \( \mathfrak{B} \) functions and proved a theorem which is a local characterization of real-valued Darboux \( \mathfrak{B} \) functions. The purpose of this paper is to generalize this theorem. To this end, the definition of a function being Darboux \( \mathfrak{B} \) at a point is given which has a metric continuum as its range. Hence, the theorem that a function is Darboux \( \mathfrak{B} \) if and only if it is Darboux \( \mathfrak{B} \) at each point.

1. Introduction. A function having domain and range a subset of the real line is said to be a Darboux function if the image of every connected subset of its domain is a connected set. A detail account of Darboux functions can be found in the survey article by Bruckner and Ceder [2].

A. M. Bruckner and J. B. Bruckner [1] gave the definition of Darboux \( \mathfrak{B} \) functions which are defined on a euclidean space and having a separable metric space as the range, and proved a theorem which is a local characterization of real-valued Darboux \( \mathfrak{B} \) functions. Then they made the statement that it would be of interest to know whether such a theorem exists in case the range is not a subset of the real line. The purpose of this paper is to give an answer to this query. The local condition given by Bruckner and Bruckner [1] was in terms of limit superior and limit inferior. That local condition will be replaced by one where the range of the function is a metric continuum. How much less one can assume about the range without losing the results is an open question.

§2 contains some of the needed definitions and a definition of Darboux \( \mathfrak{B} \) at a point which replaces the local condition given in [1]. §3 contains the main theorem which is preceded by a lemma.

2. Preliminaries. In this paper \( X \) will be a euclidean space, \( Y \) a metric continuum, and \( \mathfrak{B} \) a topological base for \( X \) such that each member of \( \mathfrak{B} \) is connected. A function \( f \) defined on \( X \) and having range \( Y \) is said to be
Darboux $\mathcal{B}$ provided that if $U$ is in $\mathcal{B}$, then $f(\text{cl}(U))$ is connected in $Y$. A base $\mathcal{B}$ for $X$ is said to satisfy condition (*) provided any translation of an element of $\mathcal{B}$ is in $\mathcal{B}$. The base $\mathcal{B}$ satisfies condition (**) provided that for each $x \in X$ and $U \in \mathcal{B}$, $x \in \text{cl}(U)$, there exists $V \in \mathcal{B}$ such that $x \in \text{cl}(V)$ and $\text{cl}(V) - (x) \subseteq U$.

Let $f$ be a function defined on $X$ and having range $Y$. The function $f$ is said to have property L at the point $x$ in $X$ provided that, if $U \in \mathcal{B}$ and $x$ is a boundary point of $U$, then $f(x)$ is a limit point of $f(U)$. If $x$ is a point in $X$, then a point $y$ in $Y$ is said to be a limit point of $f$ at $x$ provided that there exists a sequence $(x_n)_{n=1}^{\infty}$ of points in $X$ which converges to $x$ such that the sequence $(f(x_n))_{n=1}^{\infty}$ converges to $y$.

**Definition.** Let $X$ be a euclidean space and let $\mathcal{B}$ be a base of connected sets for $X$ which satisfies conditions (*) and (**). Let $f$ be a function defined on $X$ and having range $Y$. The function $f$ is said to be Darboux $\mathcal{B}$ at the point $p$ in $X$ provided that the following statements are true.

1. $f$ has property L at the point $p$ and $f(p)$ is a limit point of $f$ at $p$.
2. If $U \in \mathcal{B}$, $p \in U$, and $a$ and $b$ are limit points of $f$ at $p$ relative to $U$, then for each closed subset $C$ of $Y$ which separates $a$ from $b$ there exists $x \in U$ such that $f(x)$ is in $C$.

3. **A local characterization.** A lemma will be proved before proving the main theorem.

**Lemma.** Let $f$ be a function defined on $X$ and having range $Y$ and let $\mathcal{B}$ be a basis for $X$ which satisfies conditions (*) and (**). Let $U \in \mathcal{B}$ and let $f(A)$ and $f(B)$ be two mutually separated sets such that the union of $A$ and $B$ is $\text{cl}(U)$. If $f$ has property L at each point in $X$ and $P$ is the boundary of $A$ relative to $\text{cl}(U)$, then the following statements are true.

1. $P$ is the boundary of $B$ relative to $\text{cl}(U)$.
2. $P$ is closed in $X$.
3. $P \cap U$ is nonempty, dense-in-itself, and of type $G_\delta$.
4. $A \cap P'$ and $B \cap P'$ are both dense in $P' = P \cap U$.

**Proof.** It is obvious that $P$ is the boundary of $B$ relative to $\text{cl}(U)$ and $P$ is closed in $X$.

To prove that $P \cap U$ is nonempty, note that $P$ is nonempty; for, if $P$ were empty, then $A$ and $B$ would form a separation of the connected set $\text{cl}(U)$. Suppose $P \cap U$ is empty. Then $P \subseteq \text{cl}(U) - U$, and $U \subseteq A$ or $U \subseteq B$. Assume $U \subseteq A$. Choose any $x$ in $B$. Then $x$ is in $\text{cl}(U) - U$ and $x$ is a boundary point of $U$. Thus $f(x)$ is a limit point of $f(U)$. So $f(x)$ is a limit point of
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$f(A)$ which is in $f(B)$. This contradicts the fact that $f(A)$ and $f(B)$ are mutually separated. Therefore $P \cap U$ is nonempty.

Now suppose $P \cap U$ is not dense-in-itself. Then there is a point $x$ in $P \cap U$, say $x$ is in $P \cap A$, and a $V$ in $\mathcal{B}$, $V \subset U$, such that $P \cap V = \{x\}$. If $X$ is of dimension greater than or equal to 2, then $V - \{x\}$ is connected. So $V - \{x\} \subset A$ or $V - \{x\} \subset B$. Since $x$ is in $P$, $V - \{x\}$ is not a subset of $A$. So $V - \{x\} \subset B$. By condition (**), there is a $W \in \mathcal{B}$ such that $x$ is in $\text{cl}(W)$ and $\text{cl}(W) - \{x\} \subset V - \{x\} \subset B$. Since $x$ is a boundary point of $W$, $f(x)$ is a limit point of $f(W)$. So $f(x)$ is a limit point of $f(B)$ which is in $f(A)$. This is a contradiction. Thus $P \cap U$ is dense-in-itself if $X$ is of dimension greater than or equal to 2. If the dimension of $X$ is 1, the result is easily proved.

Since $P$ is closed and $U$ is open, $P \cap U$ is of type $G_6$.

To prove that $A \cap P'$ is dense in $P'$, suppose that $A \cap P'$ is not dense in $P'$. Let $V \subset U$ be an open sphere centered at a point $p \in P' \cap B$ such that $V \cap A \cap P'$ is empty. Then $A \cap V$ is open and nonempty. Let $x$ be a point of $A \cap V$ such that $d(x, X - V)$ is greater than $2d(x, p)$ where $d$ is the euclidean metric of $X$. Choose $W \in \mathcal{B}$ such that $x \in W \subset A \cap V$ and $d(\text{cl}(W), X - V)$ is greater than $d(\text{cl}(W), p)$. Let $q$ be a point of $B$ nearest $x$. Then $q$ is in $V$. Let $W'$ be a translation of $W$ such that $q \in \text{cl}(W') - W'$ and $W' \subset A$. By condition (*), $W' \in \mathcal{B}$. Now choose $W'' \in \mathcal{B}$ such that $q \in \text{cl}(W'')$ and $\text{cl}(W'') - (q) \subset W'$. Now $q \in B$ and $\text{cl}(W'') - (q) \subset A$. Since $q$ is a boundary point of $W''$, we have a contradiction by the hypothesis of the theorem. Thus $A \cap P'$ is dense in $P'$. In a similar way, it can be shown that $B \cap P'$ is dense in $P'$.

Theorem. Let $X$ be a euclidean space and let $\mathcal{B}$ be a base of connected sets for $X$ which satisfies conditions (*) and (**). Let $f$ be a function defined on $X$ and having a metric continuum $Y$ as its range. Then $f$ is Darboux $\mathcal{B}$ if and only if $f$ is Darboux $\mathcal{B}$ at each point in $X$.

Proof. Suppose $f$ is not Darboux $\mathcal{B}$. Then there exists $U \in \mathcal{B}$ such that $f(\text{cl}(U))$ is not connected. Let $f(A)$ and $f(B)$ be two mutually separated sets such that the union of $A$ and $B$ is $\text{cl}(U)$.

Let $Y(A)$ and $Y(B)$ be two disjoint open subsets of $Y$ such that $f(A) \subset Y(A)$ and $f(B) \subset Y(B)$. Let $C$ be the complement of $Y(A) \cup Y(B)$. Then $C$ separates $Y(A)$ and $Y(B)$. Let $P$ be the boundary of $A$ in $\text{cl}(U)$ and let $P' = P \cap U$. By the lemma both the sets $A \cap P'$ and $B \cap P'$ are dense in $P'$.

We now show that if $p$ is in $A \cap U$ and $W$ is an open set containing $C$, then there exists an open set $V \in \mathcal{B}$ containing $p$ such that if $x \in \text{cl}(V)$,
then \( f(x) \in Y(A) \cup W \). Suppose the condition fails for some \( p \in A \cap U \) and \( W \supseteq C \). Let \( V_1, V_2, \ldots \) be a sequence of sets each containing \( p \) such that \( V_k \in \mathcal{B} \) for each \( k = 1, 2, \ldots \) and \( \bigcap_{k=1}^{\infty} V_k = (p) \). Then for each \( k \) there exists \( x_k \in V_k \) such that \( f(x_k) \) is not in \( Y(A) \cup W \). Since \( Y(A) \cup W \) is open and \( Y \) is compact, \( Y = Y(A) \cup W \) is compact. Thus there exists a subsequence of \( (f(x_k))_{k=1}^{\infty} \) which converges to some point \( y \in Y = Y(A) \cup W \), and hence \( y \) is a limit point of \( f \) at \( p \) which is in \( Y(B) \). Since \( f(p) \in Y(A) \) and \( f(p) \) is a limit point of \( f \) at \( p \), there exists \( x \in U \) such that \( f(x) \in C \). This is a contradiction. Similarly, if \( p \in B \cap U \) and \( W \) is an open set containing \( C \), then there exists \( V \in \mathcal{B} \) containing \( p \) such that if \( x \in \text{cl}(V) \), then \( f(x) \in Y(B) \cup W \).

Let \( W_1, W_2, \ldots \) be a sequence of open subsets of \( Y \) such that \( \bigcap_{k=1}^{\infty} W_k = C \). Let \( p_1 \in A \cap P' \). Choose \( V_1 \in \mathcal{B} \) such that \( p_1 \in V_1 \), the diameter of \( V_1 < 1 \), \( \text{cl}(V_1) \subseteq U \), and if \( x \in \text{cl}(V_1) \), then \( f(x) \in Y(A) \cup W_1 \). Since \( B \cap P' \) is dense in \( P' \), there is a \( p_2 \) in \( B \cap P' \cap V_1 \). Let \( V_2 \in \mathcal{B} \) such that \( p_2 \in V_2 \), \( V_2 \subseteq V_1 \), the diameter of \( V_2 < \frac{1}{2} \), and if \( x \in \text{cl}(V_2) \), then \( f(x) \in Y(B) \cup W_2 \). Continuing in this manner, we obtain a sequence of points \( (p_k)_{k=1}^{\infty} \) and a sequence of open sets \( (V_k)_{k=1}^{\infty} \) such that for each \( k \), the following conditions are satisfied.

1. \( p_k \in V_k \cap P', V_k \in \mathcal{B}, V_{k+1} \subseteq V_k \), the diameter of \( V_k < 1/k \).
2. If \( x \in \text{cl}(V_k) \) and \( k \) is odd, then \( f(x) \in Y(A) \cup W_k \).
3. If \( x \in \text{cl}(V_k) \) and \( k \) is even, then \( f(x) \in Y(B) \cup W_k \).

Note that the set \( \bigcap_{k=1}^{\infty} \text{cl}(V_k) \) consists of a single point, say \( p \).

Since \( \text{cl}(V_1) \subseteq U \), \( p \in U \). Since \( p \in \text{cl}(V_k) \) for each \( k = 1, 2, \ldots \), it follows that \( f(p) \) is in each \( W_k \). Thus \( f(p) \in \bigcap_{k=1}^{\infty} W_k \), and so \( f(p) \in C \). This is a contradiction, and hence \( f \) is Darboux \( \mathcal{B} \).

We now assume \( f \) is Darboux \( \mathcal{B} \) and prove that \( f \) is Darboux \( \mathcal{B} \) at each point of \( X \). The fact that \( f \) has property \( L \) of the definition of Darboux \( \mathcal{B} \) at each point is obvious. We now prove that \( f(p) \) is a limit point of \( f \) at \( p \).

Let \( (W_k)_{k=1}^{\infty} \) be a descending sequence of open subsets of \( Y \) each containing \( f(p) \) such that \( \bigcap_{k=1}^{\infty} W_k = (f(p)) \). Let \( (V_k)_{k=1}^{\infty} \) be a descending sequence of sets in \( \mathcal{B} \) each containing \( p \) such that \( \bigcap_{k=1}^{\infty} V_k = (p) \). Since \( f \) is Darboux \( \mathcal{B} \) and \( p \in V_k \), \( f(V_k) \) is a connected subset of \( Y \) which contains \( f(p) \) for each \( k = 1, 2, \ldots \). Now there exists \( x_k \in V_k \) such that \( x_k \neq p \) and \( f(x_k) \in W_k \) for each \( k = 1, 2, \ldots \). The sequence \( (x_k)_{k=1}^{\infty} \) converges to \( p \) and the sequence \( (f(x_k))_{k=1}^{\infty} \) converges to \( f(p) \). Therefore \( f(p) \) is a limit point of \( f \) at \( p \).

We now prove part (2) of the definition of Darboux \( \mathcal{B} \) at a point. Let \( U \)
\( \in \mathcal{B}, p \in U, \) and let \( a \) and \( b \) be limit points of \( f \) at \( p \) relative to \( U \). Let \( C \) be a closed subset of \( Y \) which separates \( a \) from \( b \). Thus there exists two mutually separated open sets \( V \) and \( W \) such that \( Y - C = V \cup W, a \in V \) and \( b \in W \). Since \( a \) and \( b \) are limit points of \( f \) at \( p \), there exist \( y, z \in U \) such that \( f(y) \in V \) and \( f(z) \in W \). Thus \( C \) separates \( f(y) \) and \( f(z) \). Since \( f(U) \) is connected, there exists \( x \in U \) such that \( f(x) \in C \), and we are finished.

REFERENCES