PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 49, Number 2, June 1975

A CONVERSE STEINHAUS THEOREM FOR LOCALLY COMPACT GROUPS

S. M. SIMMONS

ABSTRACT. Singular and absolutely continuous regular measures on locally compact groups are characterized in terms of the algebraic and topological structure of sets assigned positive measure.

A classical theorem of H. Steinhaus asserts that if a set E has positive left Haar measure, then the set EE^{-1} contains a neighborhood of the identity. In this paper we prove a strong converse to this theorem, and so obtain a new characterization of absolutely continuous measures, which does not refer to Haar measure. We then give a related characterization of purely singular measures. The construction which underlies these results is carried out in the following.

Lemma. *Let μ be a positive, finite, regular Borel measure on a locally compact group G, which is singular with respect to left Haar measure, λ, on G. Let μ be concentrated on a Borel set $R \subset G$ with $\lambda(R) = 0$. Then for each compact set $K \subset R$, there is a Borel set $P \subset K$ such that PP^{-1} has void interior and the complement of P in K is μ-null.*

Proof. Since the compact set K is contained in a compactly-generated closed and open subgroup, we may suppose without loss of generality that G is compactly generated. Since $\lambda(K) = 0$, regularity of μ allows us to choose a sequence U_i of open sets with $K \subset U_i$, for all $i = 1, 2, 3, \cdots$, and $\operatorname{Lim} \lambda(U_i) = 0$. We also choose symmetric neighborhoods W_i of the identity e, with $KW_i \subset U_i$ for all i. Because of the metrization theorem for groups [1], there exists a compact normal subgroup N of G such that G/N is separable and metric, and with $N \subset \bigcap_{i=1}^{\infty} W_i$, so that $KN \subset KW_i \subset U_i$ for all i, and consequently, $\lambda(KN) = 0$. Now let $Z = \{x \in G: \mu(xKN) = 0\}$. We claim that the cosets $\{xN: x \in Z\}$ are dense in G/N. For consider an arbitrary nonvoid open set $V \subset G$ and the corresponding open set $\{vN: v \in V\}$.

Presented to the Society, April 22, 1972; received by the editors March 4, 1974.
AMS (MOS) subject classifications (1970). Primary 43A05, 28A70.
Key words and phrases. Absolute continuity, regularity of measures.

Letting ξ_E denote the characteristic function of a set $E \subset G$, and using Fubini's theorem, we compute that

$$\int_{VN} \mu(xKN)d\lambda(x) = \int \xi_{VN}(x) \int \xi_{xKN}(y)d\mu(y)d\lambda(x)$$

$$= \int\int \xi_{VN}(x)\xi_{(KN)^{-1}}(y^{-1}x)d\lambda(x)d\mu(y)$$

$$= \int\int \xi_{VN}(yx)\xi_{(KN)^{-1}}(x)d\lambda(x)d\mu(y)$$

$$= \int \xi_{(KN)^{-1}}(x) \int \xi_{VN}(yx)d\mu(y)d\lambda(x) = 0,$$

since $\lambda((KN)^{-1}) = \lambda(KN) = 0$. Thus the original integrand, $\mu(xKN)$, will vanish for λ-almost all $x \in VN$. Taking any $x \in VN$ with $\mu(xKN) = 0$, it follows that $xN \in \{vN: v \in V\}$, so the claim is established.

Now consider the set

$$Z_1 = \{x \in G: \mu(KN \cap xKN) = 0\}.$$

Since $\mu(KN \cap xKN) \leq \mu(xKN)$, we have that $Z \subset Z_1$, so that the set $\{xN: x \in Z_1\}$ is dense in G/N. Using separability of G/N, choose a countable set $D \subset Z_1$ such that $\{dN: d \in D\}$ is dense in G/N. Now let

$$S = \bigcup \{KN \cap dKN: d \in D\}$$

and note that $S \subset KN$ and that

$$\mu(S) \leq \sum_{d \in D} \mu(KN \cap dKN) = 0.$$

Finally, let $B = KN \sim S$, the complement of S in KN. We will show that BB^{-1} has void interior. For, if there exists a nonvoid open set $V \subset BB^{-1}$, then by the choice of D, there is a $d \in D$ with $dN \in \{vN: v \in V\}$. Thus $dN \subset VN \subset BB^{-1}N$, and so there exist $k_1n_1 \in B$ and $k_2n_2 \in B$ and n_0, $n_3 \in N$ such that $dn_0 = k_1n_1(k_2n_2)^{-1}n_3$, so, using normality of N,

$$k_1n_1 = dn_0n_3^{-1}k_2n_2 \in dNN^{-1}KN = dKN.$$

But this means $k_1n_1 \in KN \cap dKN \subset S$, a contradiction, since S is disjoint from B, and k_1n_1 was chosen from B. The contradiction shows BB^{-1} has void interior. Moreover, since $K \subset KN$, we have $K \sim S \subset KN \sim S = B$, and the inclusion

$$(K \sim S)(K \sim S)^{-1} \subset B \cdot B^{-1}$$

shows that the interior of $(K \sim S)(K \sim S)^{-1}$ must also be void.

We may now take $P = K \sim S$ to establish the Lemma, noting that the complement of P in K is the set $K \cap S$, which is μ-null since S is μ-null.

It is now possible to derive the two characterization theorems.

Theorem 1. *Let μ be a complex regular Borel measure on G, with total variation $|\mu|$. The following statements are equivalent:*

(1) *For each compact E in G with $|\mu|(E) > 0$, the set EE^{-1} has nonvoid interior.*

(2) *For each compact set E in G with $|\mu|(E) > 0$, the set EE^{-1} contains a neighborhood of the identity.*

(3) *For each pair A, B of compact sets with $|\mu|(A) > 0$ and $|\mu|(B) > 0$, the set AB^{-1} has nonvoid interior.*

(4) *The measure μ is absolutely continuous with respect to left Haar measure λ on G.*

Proof. Statements (3) and (2) clearly imply (1), and that (4) implies (3) and (2) is the content of the classic Steinhaus theorem [1, Theorem 20.17]. To establish the equivalence of all four statements, we need only show that (1) implies (4). So suppose that μ satisfies the hypothesis of (1), and let $\mu = \sigma + \eta$ be the Lebesgue decomposition of μ, where σ is absolutely continuous with respect to λ, and η is singular with respect to λ. We will prove (4) by showing that $\eta = 0$. Since statement (4) is known to imply (2), and consequently (1), the measure σ satisfies the hypothesis of (1) as does μ. It follows easily that the difference, $\eta = \mu - \sigma$, also satisfies the hypothesis of (1). Suppose, for a contradiction, that $\eta \neq 0$. Then $|\eta|$ is a nontrivial measure, concentrated on a Borel set R with $\lambda(R) = 0$. By regularity of $|\eta|$, we may choose a compact set $K_1 \subset R$ with $|\eta|(K_1) > 0$. The Lemma above then guarantees the existence of a Borel set $P \subset K_1$ with $|\eta|(P) = |\eta|(K_1)$, and such that $P \cdot P^{-1}$ has void interior. Again by regularity of $|\eta|$, there is a compact set $K \subset P$ with $|\eta|(K) > 0$. But this contradicts the fact that η satisfies the hypothesis of (1), since $KK^{-1} \subset PP^{-1}$ implies that KK^{-1} has void interior. The contradiction shows that $\eta = 0$, and the proof is complete.

Our next theorem shows that singular measures have antithetical properties to those in Theorem 1.

Theorem 2. *A regular measure μ on G is singular with respect to Haar measure λ, if and only if μ is concentrated on a σ-compact set B such that BB^{-1} has void interior.*

Proof. For the "if" statement, let $\mu = \sigma + \eta$ be the Lebesgue decomposition of μ, as above. Since μ is concentrated on B, so is σ, but by the property of B in the hypothesis and Theorem 1 above, it follows that $|\sigma|(B) = 0$, so that $\sigma = 0$. Hence $\mu = \eta$, so that μ is singular.

For the "only if" part, we may suppose without loss of generality that μ is nonnegative and concentrated on E with $\lambda(E) = 0$. The conclusion will follow from the construction of a sequence $\{F_i\}$ of compact sets with the properties:

(1) $F_1 \subset F_2 \subset F_3 \subset \cdots \subset E$.

(2) For each i, $F_iF_i^{-1}$ has void interior.

(3) $\operatorname{Lim} \mu(F_i) = \mu(E)$.

This sequence is formed as follows: Using regularity of μ we choose a sequence K_i of compact sets so that $K_1 \subset K_2 \subset K_3 \subset \cdots \subset E$ and $\operatorname{Lim} \mu(K_i) = \mu(E)$. By the Lemma above, we decompose each K_i into $K_i = P_i \cup N_i$ where $P_iP_i^{-1}$ has void interior and $\mu(N_i) = 0$. Let $N = \bigcup_{i=1}^{\infty} N_i$, so that $K_1 \sim N \subset K_2 \sim N \subset \cdots \subset E$ and $\mu(K_i \sim N) = \mu(K_i)$, for all i. Again using regularity of μ, we let $F_0 = \emptyset$ and choose by induction a sequence $\{F_i\}$ of compact sets such that $F_{i-1} \subset F_i \subset K_i \sim N$, and

$$|\mu(K_i \sim N) - \mu(F_i)| = |\mu(K_i) - \mu(F_i)| \leq 1/i.$$

This being done, properties (1) and (3) are evident, and (2) follows from the inclusion $F_i \subset K_i \sim N \subset K_i \sim N_i \subset P_i$, so that $F_iF_i^{-1} \subset P_iP_i^{-1}$; the choice of P_i insures that $F_iF_i^{-1}$ has void interior.

To obtain the theorem, we let $B = \bigcup_{i=1}^{\infty} F_i$. Clearly $B \subset E$ with $\mu(E \sim B) = 0$, so that μ is concentrated on B. To see that BB^{-1} has void interior we briefly suppose the contrary, and note that because of property (1), $BB^{-1} \subset \bigcup_{i=1}^{\infty} F_iF_i^{-1}$. The Baire category theorem then asserts that one of the compact sets $F_iF_i^{-1}$ has nonvoid interior, a contradiction of property (2) of the sequence $\{F_i\}$. Since B is σ-compact by construction, B has the properties claimed in the statement of the theorem.

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DEPARTMENT OF MATHEMATICS, EASTERN WASHINGTON STATE COLLEGE, CHENEY, WASHINGTON 99004