A NOTE ON $C_c(X)$

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ABSTRACT. When $X$ is a locally convex topological linear space, the function algebra $C_c(X)$ (with continuous convergence) can have a closure operator which has infinitely many distinct iterations. The reverse situation is also possible: $X$ can be a locally compact $c$-embedded convergence space whose closure operator has infinitely many distinct iterations, whereas $C_c(X)$ is a topological space.

1. The study of $C_c(X)$, the function algebra of all real-valued continuous functions on a convergence space $X$, equipped with the continuous convergence structure, was initiated by Cook and Fischer [4] and has been further developed by subsequent investigators. Arens [1] showed in 1946 that, for a completely regular topological space $X$, $C_c(X)$ is topological iff $X$ is locally compact. The same theorem has been established [6, Theorem 3.6] in the more general case where $X$ is an $\omega$-regular convergence space. Consequently $X$ can be topological when $C_c(X)$ is not, and $C_c(X)$ can be topological when $X$ is not.

This note is concerned with the following question: If $X$ is a topology, then how far can $C_c(X)$ deviate from being a topology, and vice versa. Convergence spaces do not, in general, have idempotent closure operators, and the number of distinct iterations of the closure operator of a convergence space gives an indication of how nontopological the space is. We will refer to a convergence space whose closure operator has infinitely many distinct iterations as being highly nontopological.

A convergence space is $c$-embedded if the natural map from $X$ into $C_cC_c(X)$ is a homeomorphism, in which case $X$ is homeomorphic to Hom$_cC_c(X)$ (by [3, Theorem 1]). A convergence space is said to be locally compact if every convergent filter contains a compact set.

Theorem 1. There is a highly nontopological, locally compact, $c$-embedded convergence space $X$ such that $C_c(X)$ is a topological space.

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Theorem 2. There is a locally convex topological linear space $X$ such that $C_c(X)$ is a highly nontopological convergence space.

These theorems both follow from one example, the space $X_0$ constructed in §3. Their proofs will be deferred until §4.

2. For notation, terminology, and background information, see [2] or [3]. The term space will always mean convergence space. A space $X$ is regular if $\mathcal{F} \to x$ implies that $\text{cl}_X\mathcal{F} \to x$ ("cl" is the closure operator for $X$; $\mathcal{F}$ a filter on $X$); $X$ is $\omega$-regular if $\mathcal{F} \to x$ implies $\text{cl}_{\omega X}\mathcal{F} \to x$ (where $\omega X$ is the finest completely regular space coarser than $X$ on the same underlying set). A pretopological space (sometimes called principal space or neighborhood space) is one in which the neighborhood filter for each point $x$ converges to $x$; $X$ is pseudo-topological if $\mathcal{F} \to x$ whenever each ultrafilter finer than $\mathcal{F}$ converges to $x$. The initial lemma is Theorem 2.4 of [6].

Lemma 1. A space $X$ is $c$-embedded iff $X$ is Hausdorff, $\omega$-regular, and pseudo-topological.

For any space $X$, let $\lambda X$ denote the topological modification of $X$; that is, $\lambda X$ is the finest topological space on the same underlying set which is coarser than $X$.

Lemma 2. Let $X$ be a Hausdorff, locally compact, regular, pseudo-topological space such that $\lambda X$ is completely regular. Then $X$ is $c$-embedded.

Proof. By Lemma 1, it is sufficient to show that $X$ is $\omega$-regular. From [5, Corollary 2.4] it follows that a compact, regular, Hausdorff space has an idempotent closure operator. Thus, if $\mathcal{F}$ is a convergent filter on $X$, $\text{cl}_X\mathcal{F}$ must have a filter base of compact sets, and so $\text{cl}_X\mathcal{F} = \text{cl}_{\lambda X}\mathcal{F}$. But, by assumption, $\text{cl}_{\lambda X}\mathcal{F} = \text{cl}_{\omega X}\mathcal{F}$, and so $X$ is $\omega$-regular.

The next lemma describes a method for constructing a base of $\lambda X$-open neighborhoods for a point in a pretopological space. The straightforward proof is omitted.

Let $N$ denote the set of natural numbers, including 0.

Lemma 3. In a pretopological space $X$, the $\lambda X$-neighborhood filter at a point $x$ has a filter base of sets of the form $\bigcup \{V_n : n \in N\}$, where $V_0$ is an $X$-neighborhood of $x$, and $V_n$ is defined recursively as a union of $X$-neighborhoods of points $z$ in $V_{n-1}$.

If $X$ is a linear space with a convergence structure relative to which the linear operations are continuous, then $X$ is called a convergence linear
space. If $X$ is a convergence linear space, and each filter $F$ converging to $x$ contains a filter $\mathcal{G}$ converging to $x$ which has a filter base of convex sets, then $X$ is said to be locally convex. A detailed discussion of these concepts can be found in [7].

**Lemma 4.** For any space $X$, $C_c(X)$ is a locally convex convergence linear space.

**Proof.** It is well known that $C_c(X)$ is a convergence linear space, so we will show only that $C_c(X)$ is locally convex. Recall that a filter $\mathcal{A}$ converges to $f$ in $C_c(X)$ iff $\mathcal{A}(F) \to f(x)$ on the real line whenever $F \to x$ in $X$. Assume $\mathcal{A} \to f$, and let $\mathcal{A}_0$ be the filter on $C_c(X)$ generated by the convex hulls of members of $\mathcal{A}$. Let $F \to x$ on $X$, and let $W$ be a closed, convex neighborhood of $f(x)$ on the real line. Choose $A \in \mathcal{A}$, $F \in \mathcal{F}$ such that $A(F) \subset W$. If $A_0$ is the convex hull of $A$, then it follows easily that $A_0(F) \subset W$. Thus $\mathcal{A}_0 \to f$, and $X$ is locally convex.

3. We shall now construct a locally compact, $c$-embedded, pretopological space $X_0$ whose closure operator has infinitely many distinct iterations. For each $n \in \mathbb{N}$, let $S^n = \{x^n_j: j \in \mathbb{N}\}$ be a set of distinct elements, and assume $S^n_i \cap S^n_j$ is empty if $i \neq j$. Let $X_0 = \bigcup_{n \in \mathbb{N}} S^n$. Furthermore, for each $n \in \mathbb{N}$, let $S^n_j$ be partitioned into infinitely many disjoint sets $\{S^n_k: k \in \mathbb{N}\}$. Of course, an element $x^n_j$ of $S^n$ must be in some member $S^n_k$ of the partition, but no relationship between $j$ and $k$ is assumed.

A pretopology will be constructed on the set $X_0$ by defining the neighborhood filter for each $x$ in $X_0$ as follows:

1. $\mathcal{N}(x^0_j)$ is the ultrafilter generated by $\{x^0_j\}$, all $j$ in $\mathbb{N}$;
2. For $i \geq 1$ and $j$ in $\mathbb{N}$, $\mathcal{N}(x^i_j)$ is the filter generated by all sets of the form $(S^{i-1}_j - F) \cup \{x^i_j\}$, where $F$ is any finite subset of $X_0$. In the discussion that follows, we will refer to $\{x^0_j\}$ as the basic $X_0$-neighborhood of $x^0_j$, for all $j$ in $\mathbb{N}$, and we will refer to a set of the form $(S^{i-1}_j - F) \cup \{x^i_j\}$ as described in (2) as a basic $X_0$-neighborhood of $x^i_j$, for all $i \geq 1$ and $j$ in $\mathbb{N}$.

Note that basic neighborhoods of distinct points are disjoint.

By our construction, $X_0$ is a Hausdorff pretopological (hence, pseudotopological) space. Note that a basic neighborhood of any point in $X_0$ is compact (hence, closed), and it follows that $X_0$ is locally compact and regular. To show that $X_0$ is $c$-embedded, it remains, in view of Lemma 2, to show that $\lambda X$ is completely regular. This will be accomplished by showing that, for any point $x^i_j$ in $X_0$, that $x^i_j$ has a base of $\lambda X$-neighborhoods which are both open and closed.
For each \( i, j \in \mathbb{N} \), let \( W_j^i \) be a \( \lambda X_0 \)-neighborhood of \( x_j^i \) constructed in the manner described in Lemma 3, where each \( X_0 \)-neighborhood of a point \( x_m^k \) used in this construction is a basic \( X_0 \)-neighborhood of \( x_m^k \). It follows from Lemma 3 that \( W_j^i \) is open for all \( i, j \in \mathbb{N} \); we will next show that each such set is also closed.

For arbitrarily chosen indices \( i \) and \( j \), assume that an ultrafilter \( \mathcal{G} \) containing \( W_j^i \) converges to a point \( x_n^m \). By the construction of \( X_0 \), exactly one of the following assertions must be true: (a) \( \mathcal{G} \) is the ultrafilter generated by \( \{ x_n^m \} \) (in which case \( x_n^m \in W_j^i \)); (b) \( \mathcal{G} \) contains \( S_n^m \). In the latter case, one of the basic \( X_0 \)-neighborhoods involved in the construction of \( W_j^i \) must be of the form \( S_n^m - F \), where \( F \) is finite, and this can happen only if \( x_n^m \) itself is in \( W_j^i \). Thus \( W_j^i \) is both open and closed. Since sets of the form \( W_j^i \) form a base for the \( \lambda X_0 \)-neighborhood filter at \( x_j^i \), \( \lambda X_0 \) is zero-dimensional, and hence completely regular.

We have now established, usingLemma 2, that \( X_0 \) is c-embedded and locally compact. It remains to show that \( X_0 \) is highly non-topological. Note that \( \text{cl}_{X_0} S^n_0 = S^n_0 \cup S_1 \), and \( \text{cl}_{X_0} S^n_0 = \bigcup_{k \leq n} S^k_0 \). Thus the closure operator for \( X_0 \) has infinitely many distinct iterations.

4. It remains to prove the two theorems. To prove Theorem 1, we make use of Theorem 3.2 [6] which asserts that when \( X \) is locally compact, \( C_c(X) \) is a topology; thus Theorem 1 is established by taking \( X = X_0 \).

To prove Theorem 2, let \( X = C_c(X_0) \); by Lemma 4, \( X \) is locally convex, and so, by our preceding observations, a locally convex topological linear space. Since \( X_0 \) is c-embedded, \( X_0 \) is homeomorphic to \( \text{Hom}_c C_c(X_0) \). But \( \text{Hom}_c C_c(X_0) \) is a closed subspace of \( C_c C_c(X_0) = C_c(X) \), and it follows from [5, Corollary 1.4] that the closure operator for \( C_c(X) \) has at least as many distinct iterations as does the closure operator of \( \text{Hom}_c C_c(X_0) \) (i.e., infinitely many). Thus \( C_c(X) \) is highly non-topological.

The fact that a locally compact, c-embedded space can be highly non-topological would appear to be of some interest in itself. Another corollary of some interest is the fact (which follows from Lemma 4 and the proof of Theorem 2) that a locally convex linear convergence space can be highly non-topological.

REFERENCES


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