PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 49, Number 2, June 1975

A NOTE ON $C_c(X)$

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ABSTRACT. When X is a locally convex topological linear space, the function algebra $C_c(X)$ (with continuous convergence) can have a closure operator which has infinitely many distinct iterations. The reverse situation is also possible: X can be a locally compact c-embedded convergence space whose closure operator has infinitely many distinct iterations, whereas $C_c(X)$ is a topological space.

1. The study of $C_c(X)$, the function algebra of all real-valued continuous functions on a convergence space X, equipped with the continuous convergence structure, was initiated by Cook and Fischer [4] and has been further developed by subsequent investigators. Arens [1] showed in 1946 that, for a completely regular topological space X, $C_c(X)$ is topological iff X is locally compact. The same theorem has been established [6, Theorem 3.6] in the more general case where X is an ω-regular convergence space. Consequently X can be topological when $C_c(X)$ is not, and $C_c(X)$ can be topological when X is not.

This note is concerned with the following question: If X is a topology, then how far can $C_c(X)$ deviate from being a topology, and vice versa. Convergence spaces do not, in general, have idempotent closure operators, and the number of distinct iterations of the closure operator of a convergence space gives an indication of how nontopological the space is. We will refer to a convergence space whose closure operator has infinitely many distinct iterations as being *highly nontopological.*

A convergence space is *c-embedded* if the natural map from X into $C_cC_c(X)$ is a homeomorphism, in which case X is homeomorphic to $\mathrm{Hom}_c C_c(X)$ (by [3, Theorem 1]). A convergence space is said to be *locally compact* if every convergent filter contains a compact set.

Theorem 1. *There is a highly nontopological, locally compact, c-embedded convergence space X such that $C_c(X)$ is a topological space.*

Received by the editors April 10, 1974.
AMS (MOS) subject classifications (1970). Primary 54A20, 54A05, 54G20.
Key words and phrases. Convergence space, continuous convergence, c-embedded space, highly nontopological space.

Theorem 2. *There is a locally convex topological linear space* X *such that* $C_c(X)$ *is a highly nontopological convergence space.*

These theorems both follow from one example, the space X_0 constructed in §3. Their proofs will be deferred until §4.

2. For notation, terminology, and background information, see [2] or [3]. The term *space* will always mean convergence space. A space X is *regular* if $\mathcal{F} \to x$ implies that $\mathrm{cl}_X \mathcal{F} \to x$ ("cl_X" is the closure operator for X; $\mathcal{F}$ a filter on X); X is ω*-regular* if $\mathcal{F} \to x$ implies $\mathrm{cl}_{\omega X}\mathcal{F} \to x$ (where ωX is the finest completely regular space coarser than X on the same underlying set). A *pretopological space* (sometimes called *principal space* or *neighborhood space*) is one in which the neighborhood filter for each point x converges to x; X is *pseudo-topological* if $\mathcal{F} \to x$ whenever each ultrafilter finer than $\mathcal{F}$ converges to x. The initial lemma is Theorem 2.4 of [6].

Lemma 1. *A space* X *is c-embedded iff* X *is Hausdorff,* ω*-regular, and pseudo-topological.*

For any space X, let λX denote the *topological modification* of X; that is, λX is the finest topological space on the same underlying set which is coarser than X.

Lemma 2. *Let* X *be a Hausdorff, locally compact, regular, pseudo-topological space such that* λX *is completely regular. Then* X *is c-embedded.*

Proof. By Lemma 1, it is sufficient to show that X is ω-regular. From [5, Corollary 2.4] it follows that a compact, regular, Hausdorff space has an idempotent closure operator. Thus, if $\mathcal{F}$ is a convergent filter on X, $\mathrm{cl}_X \mathcal{F}$ must have a filter base of compact sets, and so $\mathrm{cl}_X \mathcal{F} = \mathrm{cl}_{\lambda X} \mathcal{F}$. But, by assumption, $\mathrm{cl}_{\lambda X} \mathcal{F} = \mathrm{cl}_{\omega X} \mathcal{F}$, and so X is ω-regular.

The next lemma describes a method for constructing a base of λX-open neighborhoods for a point in a pretopological space. The straightforward proof is omitted.

Let N denote the set of natural numbers, including 0.

Lemma 3. *In a pretopological space* X, *the* λX*-neighborhood filter at a point* x *has a filter base of sets of the form* $\bigcup\{V_n : n \in N\}$, *where* V_0 *is an* X*-neighborhood of* x, *and* V_n *is defined recursively as a union of* X*-neighborhoods of points* z *in* V_{n-1}.

If X is a linear space with a convergence structure relative to which the linear operations are continuous, then X is called a *convergence linear* *space*. If X is a convergence linear space, and each filter $\mathcal{F}$ converging to x contains a filter $\mathcal{G}$ converging to x which has a filter base of convex sets, then X is said to be *locally convex*. A detailed discussion of these concepts can be found in [7].

Lemma 4. *For any space X, $C_c(X)$ is a locally convex convergence linear space.*

Proof. It is well known that $C_c(X)$ is a convergence linear space, so we will show only that $C_c(X)$ is locally convex. Recall that a filter $\mathcal{A}$ converges to f in $C_c(X)$ iff $\mathcal{A}(\mathcal{F}) \to f(x)$ on the real line whenever $\mathcal{F} \to x$ in X. Assume $\mathcal{A} \to f$, and let $\mathcal{A}_0$ be the filter on $C_c(X)$ generated by the convex hulls of members of $\mathcal{A}$. Let $\mathcal{F} \to x$ on X, and let W be a closed, convex neighborhood of $f(x)$ on the real line. Choose $A \in \mathcal{A}$, $F \in \mathcal{F}$ such that $A(F) \subseteq W$. If A_0 is the convex hull of A, then it follows easily that $A_0(F) \subseteq W$. Thus $\mathcal{A}_0 \to f$, and X is locally convex.

3. We shall now construct a locally compact, c-embedded, pretopological space X_0 whose closure operator has infinitely many distinct iterations. For each $n \in N$, let $S^n = \{x_j^n: j \in N\}$ be a set of distinct elements, and assume $S^i \cap S^j$ is empty if $i \neq j$. Let $X_0 = \bigcup\{S^n: n \in N\}$. Furthermore, for each $n \in N$, let S^n be partitioned into infinitely many disjoint sets $\{S_k^n: k \in N\}$. Of course, an element x_j^n of S^n must be in some member S_k^n of the partition, but no relationship between j and k is assumed.

A pretopology will be constructed on the set X_0 by defining the neighborhood filter for each x in X_0 as follows:

(1) $\mathcal{N}(x_j^0)$ is the ultrafilter generated by $\{x_j^0\}$, all j in N;

(2) For $i \geq 1$ and j in N, $\mathcal{N}(x_j^i)$ is the filter generated by all sets of the form $(S_j^{i-1} - F) \cup \{x_j^i\}$, where F is any finite subset of X_0. In the discussion that follows, we will refer to $\{x_j^0\}$ as the *basic* X_0 *neighborhood of* x_j^0, for all j in N, and we will refer to a set of the form $(S_j^{i-1} - F) \cup \{x_j^i\}$ as described in (2) as a *basic* X_0*-neighborhood of* x_j^i, for all $i \geq 1$ and j in N. Note that basic neighborhoods of distinct points are disjoint.

By our construction, X_0 is a Hausdorff pretopological (hence, pseudotopological) space. Note that a basic neighborhood of any point in X_0 is compact (hence, closed), and it follows that X_0 is locally compact and regular. To show that X_0 is c-embedded, it remains, in view of Lemma 2, to show that λX is completely regular. This will be accomplished by showing that, for any point x_j^i in X_0, that x_j^i has a base of λX-neighborhoods which are both open and closed.

For each $i, j \in N$, let W_j^i be a λX_0-neighborhood of x_j^i constructed in the manner described in Lemma 3, where each X_0-neighborhood of a point x_m^k used in this construction is a basic X_0-neighborhood of x_m^k. It follows from Lemma 3 that W_j^i is open for all $i, j \in N$; we will next show that each such set is also closed.

For arbitrarily chosen indices i and j, assume that an ultrafilter $\mathcal{G}$ containing W_j^i converges to a point x_n^m. By the construction of X_0, exactly one of the following assertions must be true: (a) $\mathcal{G}$ is the ultrafilter generated by $\{x_n^m\}$ (in which case $x_n^m \in W_j^i$); (b) $\mathcal{G}$ contains S_n^{m-1}. In the latter case, one of the basic X_0-neighborhoods involved in the construction of W_j^i must be of the form $S_n^{m-1} - F$, where F is finite, and this can happen only if x_n^m itself is in W_j^i. Thus W_j^i is both open and closed. Since sets of the form W_j^i form a base for the λX_0-neighborhood filter at x_j^i, λX_0 is zero-dimensional, and hence completely regular.

We have now established, using Lemma 2, that X_0 is c-embedded and locally compact. It remains to show that X_0 is highly nontopological. Note that $\mathrm{cl}_{X_0} S^0 = S^0 \cup S^1$, and $\mathrm{cl}_{X_0}^n S^0 = \bigcup\{S^k \colon k \leq n\}$. Thus the closure operator for X_0 has infinitely many distinct iterations.

4. It remains to prove the two theorems. To prove Theorem 1, we make use of Theorem 3.2 [6] which asserts that when X is locally compact, $C_c(X)$ is a topology; thus Theorem 1 is established by taking $X = X_0$.

To prove Theorem 2, let $X = C_c(X_0)$; by Lemma 4, X is locally convex, and so, by our preceding observations, a locally convex topological linear space. Since X_0 is c-embedded, X_0 is homeomorphic to $\mathrm{Hom}_c C_c(X_0)$. But $\mathrm{Hom}_c C_c(X_0)$ is a closed subspace of $C_c C_c(X_0) = C_c(X)$, and it follows from [5, Corollary 1.4] that the closure operator for $C_c(X)$ has at least as many distinct iterations as does the closure operator of $\mathrm{Hom}_c C_c(X_0)$ (i.e., infinitely many). Thus $C_c(X)$ is highly nontopological.

The fact that a locally compact, c-embedded space can be highly nontopological would appear to be of some interest in itself. Another corollary of some interest is the fact (which follows from Lemma 4 and the proof of Theorem 2) that a locally convex linear convergence space can be highly nontopological.

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