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SOME UNIVERSALLY PIERCED ARCS IN E^3

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ABSTRACT. A subset X of E^3 is said to be *universally pierced* if each 2-sphere containing X can be pierced by a tame arc at each point of X. We show that an arc A is universally pierced provided A has a shrinking point p such that either p lies in a tame arc in A or $E^3 - A$ has 1-ALG at p. Applying this result we show the existence of infinitely many wild universally pierced arcs.

Introduction. A subset X of E^3 is called a *pierced set* if there exists a 2-sphere S in E^3 containing X such that S can be pierced by a tame arc at each point of X. For example, a tame arc is universally pierced [5, Theorem 5] and consequently is a pierced set. The results in this paper support the following conjecture.

Conjecture. If X is a pierced arc in E^3, then X is universally pierced.

The truth of the conjecture would mean that the property of being pierced by a tame arc at a point of the arc X is independent of the 2-sphere that contains X. In this note we give sufficient conditions, in terms of the following definitions, for an arc to be universally pierced.

A point p of a set X is said to be a *shrinking point* of X if, for every open set U containing $X - \{p\}$, there is an open set V containing $X - \{p\}$ such that each loop in $V - X$ shrinks to a point in $U - X$. For example, the Fox-Artin arc [4] X, which is wild at a single point p, has p as a shrinking point but has no other shrinking point.

The complement of an arc X in E^3 is said to have 1-*ALG* (*abelian local fundamental group*) *at a point* p of X if, for each open set U containing p, there is an open set V such that $p \in V \subset U$ and each loop in $V - X$ which bounds homologically (Z coefficients) in $U - X$ also bounds a singular disk in $U - X$. If p is an endpoint of an arc X, then this simply says that $E^3 - X$ is 1-LC at p. If the complement of an arc A in E^3 has 1-ALG at each of its

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points, then A is known to be tame [3, Theorem 3.16]. In the example mentioned above [4], the complement of the arc has 1-ALG at every point except at the shrinking point p.

The main theorems. The following lemma shows that to prove that a set X is universally pierced it suffices to consider only 2-spheres that contain X and that are locally tame modulo X. It follows from this lemma and [9] that Rosen's special arc [9] is universally pierced; however, this result also follows from Theorem 1.

Lemma 1. *If X is an arc in E^3 such that every 2-sphere containing X that is locally tame modulo X can be pierced by a tame arc at each point of X, then X is universally pierced.*

Proof. Let S be a 2-sphere containing X. (If there is no such sphere, then X vacuously satisfies the definition of "universally pierced".) Using Bing's side approximation theorem relative to the open set $S - X$ (see [2, Theorem 4.6.5]) we find two 2-spheres S_1 and S_2, each locally tame modulo X and lying (except for X and the unions H_1 and H_2 of pairwise disjoint collections of disks in S_1 and S_2, respectively) in opposite complementary domains of S. Let p be in X. By hypothesis, S_1 and S_2 can each be pierced by a tame arc at p. It follows [5, Theorem 6] that tame arcs A_1 and A_2 exist in $S_1 - H_1$ and $S_2 - H_2$, respectively, such that p is an endpoint of both A_1 and A_2 and $A_1 \cap X = A_2 \cap X = \{p\}$. Thus S is accessible by a tame arc from each of its complementary domains, and it follows from [7, Theorems 2 and 3] that S can be pierced by a tame arc at p.

Theorem 1. *If an arc A in E^3 has a shrinking point p and $E^3 - A$ has 1-ALG at p, then A is universally pierced.*

Remarks on Theorem 1. The condition "$E^3 - A$ has 1-ALG at p" cannot be removed since the Fox-Artin example [4] satisfies the other hypotheses and is clearly not universally pierced. However, if A is required to be a pierced set with a shrinking point, then A ought to be universally pierced. The arc described by Alford [1] satisfies these conditions but $E^3 - A$ does not have 1-ALG at the shrinking endpoint; hence we have no proof that Alford's arc is universally pierced.

Proof of Theorem 1. By Lemma 1 it suffices to consider a 2-sphere S containing A where S is locally tame modulo A. We first use the "$E^3 - A$ has 1-ALG" hypothesis to show that S can be pierced by a tame arc at p. For convenience we assume that p is not an endpoint of A.

Let U' be an open set containing p, and let D be a disk in $S \cap U'$ such that $p \in \operatorname{Int} D$ and $A \cap \operatorname{Bd} D$ is a two-point set. Let U be an open set containing p such that $U \cap S \subset \operatorname{Int} D$ and $U \subset U'$, and let $C = S \cup \operatorname{Int} S$. We shall exhibit an open set V containing p such that each loop in $V \cap (C - \{p\})$ is homotopic to a constant in $U' \cap (C - \{p\})$. The same argument will work for $C^* = S \cup \operatorname{Ext} S$, so it will then follow from McMillan's characterization of piercing points [7, Theorem 1] that S can be pierced by a tame arc at p. Since $E^3 - S$ is 1-ulc [10, Chapter X], there must be an open set V' such that $p \in V' \subset U$ and each loop in $V' - S$ bounds homologically in $U - S$. Now choose an open set V such that $p \in V \subset V'$ and each loop in $V - A$ that bounds homologically in $U - A$ also contracts in $U - A$.

Let l be a loop in $V \cap (C - \{p\})$. Since l can be moved with a homotopy (in $C - \{p\}$) into $\operatorname{Int} C$, we may assume that l lies in $\operatorname{Int} C$. Since l is a loop in $V' - S$, it bounds homologically in $U - S$ and consequently in $U - A$. Then by the choice of V, l contracts in $U - A$. Let E be a singular disk in $U - A$ bounded by l and notice that $E \cap S \subset \operatorname{Int} D - A$. From the construction of D we see that $E \cap S$ lies in the union D' of the interiors of two disks in $D - A$. By Theorem 4.2.2 of [2] we may adjust E to a new singular disk E' where $E' \subset E \cup D'$, $E' \subset C$, and l bounds E'. Thus l contracts in $U' \cap (C - \{p\})$, as we wished to show.

We must now show that S can be pierced at every point of A. The fact that p is a point where S can be pierced by a tame arc allows us to find a tame arc B in S such that $B \cap A = \{p\}$ [5, Theorem 6]. Let q be the endpoint of B different from p, and let G be a disk in $S - A$ such that $G \cap B = \{q\}$. We shall construct a new 2-sphere S' by removing the interior of the tame disk G from S and replacing it with a new disk F. Since G is well away from A it is clear that S and S' share the same piercing properties at A. The disk F is constructed so that it is tame except at a point x in $\operatorname{Int} F$ where it is wild (see [2, Figure 3, p. 270]), $\operatorname{Bd} F = \operatorname{Bd} G$, and $F \cap S \subset G$. Since $S' = (S - G) \cup F$, it is clear that S' cannot be pierced by a tame arc at the isolated wild point x.

Let T be the union of $A \cup B$ with an arc R in F, where R has q and x as its endpoints. Once we show that T is cellular, it will follow from [8, Theorem 3] (or from [6, Theorem 1] if F is placed in the appropriate component of $E^3 - S$) that T contains at most one nonpiercing point of S'. The theorem then follows since x is a known nonpiercing point of S'.

To show that T is cellular we let U be an open set containing T, and we will indicate how to obtain a 3-cell M in U with $T \subset \operatorname{Int} M$. Let W be a

tame ball centered at p, lying in U, whose boundary intersects B at a single point; and let V be an open set containing $A - \{p\}$ such that loops in $V - A$ shrink in $U - T$ and $V \cap (B \cup R) = \varnothing$. There is a finite collection $\{D_1, D_2, \cdots, D_n\}$ of disjoint disks on Bd W whose interiors cover $A \cap \text{Bd } W$ such that each D_i lies in V. Each Bd D_i bounds a singular disk E_i in $U - T$, so $(\text{Bd } W - \bigcup D_i) \cup (\bigcup E_i)$ is a singular 2-sphere H which does not intersect A and whose intersection with $B \cup R$ is the point of the set $B \cap \text{Bd } W$. It is easy to construct a 3-cell Y lying in U with $B \cup R$ in its interior so that $H \cap \text{Bd } Y$ is the union of three concentric simple closed curves J_1, J_2, and J_3 on Bd W. (One begins with a round ball Z centered at x such that $R \cap \text{Bd } Z$ consists of three points. Then three disks are found on Bd Z whose interiors contain $R \cap \text{Bd } Z$. These disks are then pushed along $R \cup B$ and "over" p.) We add to Bd Y − Bd W three singular disks in $H - T$ bounded by J_1, J_2, and J_3 to form a singular 2-sphere K' such that $K' \subset U$, $K' \cap T = \varnothing$, and K' separates T from $E^3 - U$. Now an application of the sphere theorem (see [2, Theorem 4.5.6]) yields a nonsingular polyhedral 2-sphere K with the same properties, and K bounds a 3-cell M containing T in its interior. Since K separates T from $E^3 - U$, we see that the 3-cell M bounded by K lies in U.

Corollary. *Rosen's special arc* [9] *is universally pierced.*

Theorem 2. *If A is an arc with a shrinking point p and p lies in a tame subarc of A, then A is universally pierced.*

Proof. If p lies in the interior of a tame arc in A, then Theorem 2 follows directly from Theorem 1. In any case, the proof of Theorem 1 gives the crucial steps. To see this, let S be a 2-sphere containing A and locally tame modulo A, and choose a tame arc B such that $A \cap B = \{p\}$. Now a Fox-Artin arc [4] F is added just as in the proof of Theorem 1, and we see that $T = A \cup B \cup R$ is cellular the same way, even though it is not necessarily an arc. Again T can contain at most the one nonpiercing point x of the sphere S'. Thus S is pierced at each point of A by a tame arc.

Applications. The results of the previous section can be used to produce infinitely many wild universally pierced arcs. We begin by considering the wild arc A described by Alford [1] which has a shrinking endpoint p (by its construction) and which lies on the boundary S of a 3-cell in E^3. It is known that p is a point where S can be pierced by a tame arc; consequently there is a tame arc B on S such that $B \cap A = \{p\}$ [5]. Although we do not

prove that Alford's arc A is universally pierced, Theorem 2 shows that the arc $A_t = A \cup B$, formed by adding a tame arc to A, is universally pierced. This construction gives a hint for the construction of infinitely many such arcs.

Theorem 3. *Alford's arc with a tame tail (described above) is universally pierced.*

Theorem 4. *There exist infinitely many inequivalent universally pierced arcs in E^3.*

Proof. Let S be a round sphere in E^3, and let X be an arc in S with endpoints x and y. Let $\{D_i\}$ be a null sequence of disjoint disks on S such that $\{D_i\}$ converges to $\{x\}$ and $D_i \cap X$ is a spanning arc X_i of D_i. On each D_i we construct an Alford arc A_i such that A_i is a spanning arc of a disk E_i near D_i where Bd E_i = Bd D_i and $E_i \cap S \subset D_i$. The construction is done so that

$$A = \left(X - \bigcup_1^\infty X_i\right) \cup \left(\bigcup_1^\infty A_i\right)$$

is an arc from x to y. For each n, let

$$B_n = \left(X - \bigcup_{i=1}^n X_i\right) \cup \left(\bigcup_{i=1}^n A_i\right).$$

Each A_i has a shrinking endpoint p_i, and we insist that A_i be constructed so as to have p_i the first point of $A_i \cap B_n$ $(i \leq n)$ when x is considered the first point of B_n. Then p_i lies in a tame arc T_i in B_n, and the arc $A_i \cup T_i$ is universally pierced by Theorem 3. Since B_n is the union of universally pierced arcs, it is also universally pierced. Of course B_n and B_m $(n \neq m)$ are not equivalently embedded in E^3 since the wild set in B_n has n components while B_m's wild set has m components.

Questions. We believe that all of the following questions have affirmative answers. Three of these questions are merely special cases of the conjecture stated earlier.

1. If an arc A is a pierced set and A is cellular, then is A universally pierced?

2. Is Alford's arc [1] universally pierced? This question is equivalent to one of Rosen's by Lemma 1.

3. If an arc A is a pierced set and A has a shrinking point, then is A universally pierced?

4. Is every pierced nondegenerate continuum universally pierced?

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