PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 49, Number 2, June 1975

FREE S^3-ACTIONS ON SIMPLY CONNECTED EIGHT-MANIFOLDS[1]

RICHARD I. RESCH

ABSTRACT. In this paper the canonical equivalence between free actions of a compact Lie group G and principal G-bundles is used to apply the theory of fiber bundles to the problem of classifying free differentiable S^3-actions. The orbit spaces that may occur are determined and a calculation of homotopy classes of maps from these spaces into the classifying space for principal S^3-bundles is made with the aid of the Postnikov system for S^4. The bundles corresponding to these classes of maps are then studied to prove that for each positive integer k there exist exactly three simply connected 8-manifolds which admit free differentiable S^3-actions and have second homology group free of rank k, and that the action on each of these manifolds is unique. It is also proved that even if the second homology group of the 8-manifold has torsion, it can admit at most one action.

1. **Introduction.** Free differentiable S^1-actions on simply connected 6-manifolds were studied by Goldstein and Lininger in [3], [4] and [6]. It follows from their work that such manifolds can admit at most two distinct actions. In this paper we show that if a simply connected 8-manifold M admits a free differentiable S^3-action, then that action is unique.

Let η be the nontrivial S^3-bundle over S^2, let N_k be the connected sum of k copies of $S^2 \times S^3$ and let N_k' be the connected sum of η with $(k-1)$ copies of $S^2 \times S^3$. The main result in the torsion-free case is the following.

Theorem 3. *For each positive integer k, there exist exactly three simply connected* 8*-manifolds which admit free differentiable S^3-actions and have second homology group free of rank k. These manifolds are* $S^3 \times N_k$,

Presented to the Society, January 23, 1973 under the title *The classification of free differentiable S^3-actions on eight and nine dimensional manifolds*; received by the editors July 3, 1973 and, in revised form, February 26, 1974.

AMS (MOS) subject classifications (1970). Primary 57E25; Secondary 55F15.

Key words and phrases. Free S^3-action, principal S^3-bundle, spin manifold, Postnikov system, second Stiefel-Whitney class, Sq^2.

[1] This paper represents part of the author's Ph.D. dissertation prepared at the University of Connecticut under the direction of Professor Soon-Kyu Kim.

$S^3 \times N_k'$ and the nontrivial S^3-bundle over N_k. Furthermore, there is a unique free S^3-action on each of these manifolds.

2. **Finiteness of the number of actions.** Let M denote a closed simply connected differentiable 8-manifold which adimts a free differentiable S^3-action with orbit space N.

Lemma 1. *If N is a spin manifold, then there are two principal S^3-bundles over N; otherwise, there is just one principal S^3-bundle over N.*

Proof. It is sufficient to calculate homotopy classes of maps of N into the classifying space for principal S^3-bundles, HP^∞. Since $\dim N = 5$,

$$[N, HP^\infty] \cong [N, HP^1] \cong [N, S^4].$$

We use the Postnikov system for S^4 as in [7, p. 140], obtaining the following diagram:

$$\begin{tikzcd} K(Z, 3) \arrow[r, "\Omega \mathrm{Sq}^2"] & K(Z_2, 5) \arrow[r, "i"] & X_5 \arrow[d, "p"] & \\ & & K(Z, 4) \arrow[r, "\mathrm{Sq}^2"] & K(Z_2, 6). \end{tikzcd} \tag{1}$$

The map i is the inclusion map of the fiber $K(Z_2, 5)$, p is the fiber space projection and $\Omega \mathrm{Sq}^2 = \mathrm{Sq}^2$ since the following diagram commutes:

$$\begin{tikzcd} K(Z, 3) \arrow[r, "\mathrm{Sq}^2"] \arrow[d, equal] & K(Z_2, 5) \arrow[d, equal] \\ \Omega K(Z, 4) \arrow[r, "\Omega \mathrm{Sq}^2"] & \Omega K(Z_2, 6). \end{tikzcd}$$

Next we map N into successive terms of (1) and use the fact that $[K, K(\pi, n)]$ is in one-to-one correspondence with $H^n(K; \pi)$ to obtain the following exact sequence.

$$H^3(N; Z) \xrightarrow{\mathrm{Sq}^2} H^5(N; Z_2) \xrightarrow{i} [N, X_5] \xrightarrow{p^\cdot} H^4(N; Z) \xrightarrow{\mathrm{Sq}^2} H^6(N; Z_2). \tag{2}$$

We have used the symbols Sq^2, i and p to denote the induced maps in (2) as well as the original maps in (1). N is simply connected, so $H^4(N; Z) \cong 0$ and (2) reduces to

$$0 \to H^5(N; Z_2)/\mathrm{Sq}^2(H^3(N; Z)) \to [N, X_5] \to 0.$$

Since $H^5(N; Z_2) \cong Z_2$, $[N, X_5] \cong 0$ or Z_2 depending upon whether $\mathrm{Sq}^2 \neq$

0 or $\mathrm{Sq}^2 = 0$, respectively. Now if N is a spin manifold, then $\mathrm{Sq}^2 = 0$; otherwise, $\mathrm{Sq}^2 \neq 0$, since Sq^2 is determined by the second Stiefel-Whitney class. It follows from the definition of a Postnikov system for S^4 that $[N, X_5]$ and $[N, S^4]$ are in one-to-one correspondence, so the lemma is proved. □

We can now prove that M can admit only finitely many actions by applying Barden's classification of simply connected 5-manifolds [1] to the orbit spaces N.

Theorem 1. *A closed simply connected differentiable 8-manifold M can admit only finitely many distinct free differentiable S^3-actions.*

Proof. By Lemma 1 there are at most two actions on M with a given orbit space N. Thus, the proof will be complete once we have shown that there are only finitely many possible orbit spaces. Since N is simply connected, $H^2(N; Z_2) \cong \mathrm{Hom}(H_2(N), Z_2)$, so we may consider $w_2(N)$ in $H^2(N; Z_2)$ as a homomorphism of $H_2(N)$ into Z_2 and we may arrange it, by choosing a suitable basis for $H_2(N)$, so that w_2 is nonzero on at most one generator of $H_2(N)$ [1, Lemma C]. Since w_2 maps into Z_2, this generator will have infinite order or order 2^i for some nonnegative integer i. In [1, Theorem 2.3] Barden proves that $H_2(N)$ and i provide a complete set of diffeomorphism invariants for simply connected 5-manifolds and that i depends only on w_2. Thus, $H_2(N)$ and i determine N up to diffeomorphism. By the homotopy sequence of the bundle $S^3 \longrightarrow M \longrightarrow N$ and the Hurewicz theorem, $H_2(N) \cong H_2(M)$ which is finitely generated since M is a closed manifold. Therefore, there are only finitely many choices for the generators μ of $H_2(N)$ for which $w_2(\mu) \neq 0$. Hence, there are only finitely many possibilities for the integer i. Thus, the number of distinct simply connected 5-manifolds with $H_2(N) \cong H_2(M)$ is finite and the theorem is proved. □

3. **The torsion-free case.** We now proceed to classify the actions on manifolds with torsion-free second homology group. Suppose first that M is 2-connected. If N is the orbit space of a free S^3-action on M, then N is a 2-connected 5-manifold and, therefore, N is diffeomorphic to S^5. By Lemma 1, there are two principal S^3-bundles over S^5, and it is shown in [8, p. 139] that their total spaces are not homeomorphic. Thus, we can state the following theorem.

Theorem 2. *There exist exactly two 2-connected 8-manifolds which admit free differentiable S^3-actions. These are $S^3 \times S^5$ and the nontrivial 3-*

sphere bundle over S^5. Furthermore, there is a unique free S^3-action on each of these manifolds. □

Now suppose that $H_2(M)$ is free of rank $k > 0$. Then $H_2(N)$ is also free of rank k. Since N is simply connected, Barden's results imply that N is diffeomorphic to N_k or N'_k (see the Introduction). Since $w_2(S^2 \times S^3) = 0$ and $w_2(\eta) \neq 0$, N_k is a spin manifold and N'_k is not. Therefore, Lemma 1 implies that there are two principal S^3-bundles over N_k and only one over N'_k.

Let E^0_k and E^1_k be the total spaces of the trivial and nontrivial S^3-bundles over N_k, respectively, and let E^2_k be the total space of the trivial S^3-bundle over N'_k.

We will now present four lemmas which will prove that for each positive integer k the spaces E^i_k, $i = 0, 1, 2$, are distinct. In particular, we will show that no two of these spaces are of the same homotopy type.

Lemma 2. *E^0_k and E^1_k are spin manifolds.*

Proof. Since the orbit space N_k is a spin manifold, $w_1(N_k) = w_2(N_k) = 0$; and since $\dim N_k = 5$, the total Stiefel-Whitney class $w(N_k) = 1$. Now $S^3 \to E^i_k \to N_k$ is a differentiable principal fiber bundle for $i = 0, 1$, so according to [2, 5.2, p. 502] $w(N_k) = 1$ implies that $w(E^i_k) = 1$ for $i = 0, 1$. Thus, $w_1(E^i_k) = w_2(E^i_k) = 0$ for $i = 0, 1$ and the lemma is proved. □

Lemma 3. *E^2_k is not a spin manifold.*

Proof. Recall that $E^2_k = S^3 \times N'_k$ and $w_2(N'_k) \neq 0$. Let p_1 and p_2 be the projection maps onto the first and second factors of E^2_k, respectively. Then

$$w(S^3 \times N'_k) = p_1^* w(S^3) \cup p_2^* w(N'_k)$$

where p_i^* is the induced map in cohomology and $\cup$ is the cup product.

Since $w(S^3) = 1$, we have $p_1^* w(S^3) = 1$ and it follows that $w_2(S^3 \times N'_k) = p_2^* w_2(N'_k) \neq 0$. Hence, E^2_k is not a spin manifold. □

Lemmas 2 and 3 imply that the homotopy type of E^2_k is different from that of either E^0_k or E^1_k, since the Stiefel-Whitney classes of a manifold are homotopy invariants.

Next we will prove that Sq^2 distinguishes between the trivial and nontrivial 3-sphere bundles over S^5. We remark here that we have already seen (Theorem 2) that these spaces are not homeomorphic, but we shall use the present result concerning Sq^2 to show that the trivial and nontrivial 3-sphere bundles over N_k, i.e. E_k^0 and E_k^1, are not of the same homotopy type.

Denote the nontrivial 3-sphere bundle over S^5 by E. It follows from [5, p. 205] that $E = (S^3 \cup_f e^5) \cup_g e^8$, where f and g are attaching maps with $f: S^4 \to S^3$ not homotopically trivial. Then [9, Theorem 5.1, p. 89] implies that

$$\mathrm{Sq}^2: H^3(S^3 \cup_f e^5; Z_2) \to H^5(S^3 \cup_f e^5; Z_2)$$

is an isomorphism. Since $S^3 \cup_f e^5$ is the 7-skeleton of E, we conclude that $\mathrm{Sq}^2: H^3(E; Z_2) \to H^5(E; Z_2)$ is also an isomorphism.

Now consider pr: $S^3 \times S^5 \to S^3$ which induces an isomorphism in cohomology in dimension 3. Sq^2 commutes with pr*, so we have the following commutative diagram:

$$\begin{array}{ccc} Z_2 = H^3(S^3 \times S^5; Z_2) & \xrightarrow{\mathrm{Sq}^2} & H^5(S^3 \times S^5; Z_2) = Z_2 \\ \uparrow \mathrm{pr}^* & & \uparrow \mathrm{pr}^* \\ Z_2 = H^3(S^3; Z_2) & \xrightarrow{\mathrm{Sq}^2} & H^5(S^3; Z_2) = 0. \end{array}$$

Hence $\mathrm{Sq}^2: H^3(S^3 \times S^5; Z_2) \to H^5(S^3 \times S^5; Z_2)$ must be zero. Therefore, Sq^2 distinguishes between E and $S^3 \times S^5$, and we have proved the following lemma.

Lemma 4. *The trivial and nontrivial 3-sphere bundles over S^5 are distinguishable by the action of* Sq^2 *on their third cohomology groups.* □

We can now prove the following result concerning the trivial and nontrivial principal S^3-bundles over any simply connected 5-manifold which is a spin manifold.

Lemma 5. *Let N be any simply connected 5-manifold with $w_2(N) = 0$. Then the two principal S^3-bundles over N are homotopically distinct.*

Proof. Let E be the nontrivial S^3-bundle over S^5. Let $f: N \to S^5$ be any map of degree one, denote by G the induced bundle $f^!(E)$ and let b be the bundle map from G to E covering f. Let all coefficient groups be Z_2. Since $\mathrm{Sq}^2: H^3(S^3 \times N) \to H^5(S^3 \times N)$ is 0, using the Cartan formula and the fact that N is a spin manifold, it would suffice to prove that $\mathrm{Sq}^2: H^3(G) \to H^5(G)$ is nonzero.

Consider the following commutative square:

$$\begin{array}{ccc} H^3(E) & \xrightarrow{b^*} & H^3(G) \\ \Big\downarrow \mathrm{Sq}^2 & & \Big\downarrow \mathrm{Sq}^2 \\ H^5(E) & \xrightarrow{b^*} & H^5(G) \end{array}$$

If x and y are the nontrivial elements in $H^3(E)$ and $H^5(E)$, respectively, then $\mathrm{Sq}^2(x) = y$ by Lemma 4 and it would suffice to show that $b^*(y) \neq 0$.

Now consider the following commutative diagram whose rows are the Gysin cohomology sequence:

$$\begin{array}{ccccccc} H^1(S^5) & \xrightarrow{0} & H^5(S^5) & \xrightarrow{\cong} & H^5(E) & \longrightarrow & H^2(S^5) \\ \Big\downarrow & & \Big\downarrow \cong & & \Big\downarrow b^* & & \Big\downarrow \\ H^1(N) & \xrightarrow{0} & H^5(N) & \xrightarrow{1\text{-}1} & H^5(G) & \longrightarrow & H^2(N) \end{array}$$

Hence, $b^*(y) \neq 0$. □

Lemmas 1, 2, 3 and 5 imply the following theorem which completes the classification for simply connected 8-manifolds with torsion-free second homology group.

Theorem 3. *For each positive integer k, there exist exactly three simply connected 8-manifolds which admit free differentiable S^3-actions and have second homology group free of rank k. These manifolds are $S^3 \times N_k$, $S^3 \times N_k'$ and the nontrivial S^3-bundle over N_k. Furthermore, there is a unique free S^3-action on each of these manifolds.* □

4. **The torsion case.** In this section we investigate the more complicated situation in which the second homology group has torsion. The following theorem narrows the class of 8-manifolds which can admit free S^3-actions by describing the torsion part of their second homology groups.

Theorem 4. *If the simply connected 8-manifold M admits a free differentiable S^3-action, then the torsion subgroup of $H_2(M)$ is either trivial or of the form $T \oplus T$ or $T \oplus T \oplus Z_2$, for some finite abelian group T.*

Proof. We have noted that $H_2(M) \cong H_2(N)$, where N is the orbit space; and the corollary to Lemma E in [1] shows that the torsion part of the second homology group of a simply connected 5-manifold must be of this form. □

From Barden's classification of 5-manifolds (see [1] or the proof of

Theorem 1) we see that if the second homology group is specified, then the diffeomorphism class of the manifold is determined by the action of the second Stiefel-Whitney class w_2, which we consider as a homomorphism from the second homology group into Z_2. Since the second homology group of the 8-manifold determines the second homology group of the orbit space (they are isomorphic), in order to obtain an upper bound for the number of free S^3-actions on a given 8-manifold M we calculate the number of possible orbit spaces by determining the number of homomorphisms from $H_2(M)$ into Z_2.

We first use the primary decomposition of finitely generated abelian groups to write $H_2(M)$ as a direct sum of infinite cyclic groups and p-primary cyclic groups. This decomposition is unique and is a basis with the most possible elements, for if a basis has an element that is not of prime power order it can be replaced by two or more elements each of prime power order.

If h is a homomorphism between abelian groups A and B, then an h-basis for A is a basis on all of whose elements except possibly one h is zero. Lemma C [1] states that if A is a finitely generated abelian group and h is a homomorphism of A into Z_p, the cyclic group of order p, then A has a a basis with most possible elements that is also an h-basis. Thus, the basis of the primary decomposition is such a basis; and since we are concerned with homomorphisms into Z_2 we need only consider the infinite cyclic part and the 2-primary component, since the distinguished element (if one exists) can only have infinite order or order 2^i for some integer $i > 0$. The number of elements in the basis of order p^i is called the ith Ulm invariant of the p-primary component. Thus, for each Ulm invariant of the 2-primary component of $H_2(M)$ there is a homomorphism from $H_2(M)$ into Z_2. There is also the zero homomorphism; and if $H_2(M)$ has an infinite cyclic summand, then there is a homomorphism corresponding to its generator. Therefore, we have proved the following theorem.

Theorem 5. *If a simply connected* 8*-manifold* M *has* n *distinct nonzero Ulm invariants for the* 2*-primary component of* $H_2(M)$, *then* M *admits at most* $n + 2$ *free* S^3*-actions* ($n + 1$ *if* $H_2(M)$ *is a torsion group*). □

We can sharpen the result of Theorem 5 by considering the cases of spin manifolds and nonspin manifolds separately as in the following two theorems.

Theorem 6. *If a simply connected* 8*-manifold* M *that is not a spin manifold admits a free* S^3*-action, then that action is unique and* M *is diffeomorphic to* $S^3 \times N$ *for some simply connected* 5*-manifold* N *which is not a spin manifold.*

Proof. Suppose that there are two distinct free S^3-actions on M with orbit spaces N_1 and N_2. M must then be the total space of a principal S^3-bundle over each of these orbit spaces. The proofs of Lemmas 2 and 3 hold even if the homology of M has torsion, so we have that N_1 and N_2 are not spin manifolds. However, $H_2(N_1)$ and $H_2(N_2)$ are each isomorphic to $H_2(M)$.

It follows from Barden's results that if N_1 and N_2 are not diffeomorphic, but have isomorphic second homology groups, then $i(N_1) \neq i(N_2)$ (see Theorem 1). Thus, $i(S^3 \times N_1) \neq i(S^3 \times N_2)$; and since i is a homotopy invariant, $S^3 \times N_1$ is not of the same homotopy type as $S^3 \times N_2$. Not both of these manifolds can be M, so M must admit only one free S^3-action, and if the orbit space is N, then $M = S^3 \times N$. □

If M is a spin manifold, then Lemmas 2 and 3 imply that the orbit space N is a spin manifold, and so by Lemma 5 we have the following theorem.

Theorem 7. *If the simply connected* 8-*manifold* M *is a spin manifold, then it can admit at most one free* S^3-*action.* □

Thus, we conclude that if a simply connected 8-manifold admits a free S^3-action, then that action is unique.

REFERENCES

1. D. Barden, *Simply connected five-manifolds*, Ann. of Math. (2) **82** (1965), 365–385. MR **32** #1714.

2. A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces*. III, Amer. J. Math. **82** (1960), 491–504. MR **22** #11413.

3. R. Goldstein and L. Lininger, *Actions on* 2-*connected* 6-*manifolds*, Amer. J. Math. **91** (1969), 499–504. MR **39** #6350.

4. ———, *A classification of* 6-*manifolds with free* S^1-*actions*, Proc. of the Second Conference on Compact Transformation Groups, Springer-Verlag, New York, 1973.

5. I. M. James and J. H. C. Whitehead, *The homotopy theory of sphere bundles over spheres*. I, Proc. London Math. Soc. (3) **4** (1954), 196–218. MR **15**, 892.

6. L. Lininger, S^1-*actions on* 6-*manifolds*, Topology **11** (1972), 73–78.

7. R. Mosher and M. Tangora, *Cohomology operations and applications in homotopy theory*, Harper and Row, New York, 1968. MR **37** #2223.

8. N. E. Steenrod, *The topology of fibre bundles*, Princeton Math. Ser., vol. 14, Princeton Univ. Press, Princeton, N. J., 1951. MR **12**, 522.

9. ———, *Cohomology operations*, Lecture notes revised by D. Epstein, Ann. of Math. Studies, no. 50, Princeton Univ. Press, Princeton, N. J., 1962. MR **26** #3056.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF CONNECTICUT, STORRS, CONNECTICUT 06268

Current address: Division of Science and Mathematics, The College of the Virgin Islands, St. Thomas, U. S. Virgin Islands 00801