TEST MODULES
T. CHEATHAM AND R. CUMBIE

ABSTRACT. The results of this paper arose from an investigation of the class of \( \Sigma \)-modules, i.e. those modules \( M \) for which \( \text{Hom}_R(M, -) \) commutes with direct sums. A module \( T \) is called a test module if \( \text{Hom}_R(M, -) \) commutes with direct sums of copies of \( T \) only when \( M \) is a \( \Sigma \)-module. Test modules are characterized and their relation to cogenerators is investigated.

Throughout \( N \) will denote the set of natural numbers, \( R \) will denote an associative ring with identity, and module will mean unitary left \( R \)-module. For modules \( L \) and \( M \) and indexing set \( I \), \( L^{(I)} \) will denote the direct sum of \( |I| \) copies of \( L \) and, for convenience, \( \text{Hom}_R(M, L) \) will be written \( \text{Hom}(M, L) \).

The modules \( M \) for which \( \text{Hom}(M, -) \) commutes with direct sums have been called \( \Sigma \)-modules by Rentschler [5]. A systematic study of \( \Sigma \)-modules is given in his thesis [4]. \( \Sigma \)-modules have been considered by at least three other authors [1, p. 54], [2], and [3].

It follows from the definition that \( M \) is a \( \Sigma \)-module if and only if, for each family of modules \( \{L_i|i \in I\} \) and for each \( R \)-homomorphism \( f: M \to \bigoplus\{L_i|i \in I\} \), \( \pi_i f = 0 \) for all but a finite number of \( i \in I \). We will consistently use \( \pi_i: \bigoplus\{L_i|i \in I\} \to L_i \) to denote the obvious projection map. It is possible to place certain restrictions on the families \( \{L_i|i \in I\} \) which must be considered. It is only necessary to consider families, each of whose members is an injective module; the indexing set \( I \) may be taken to be countable. The following theorem gives a further reduction which is useful.

**Theorem 1.** A module \( M \) is a \( \Sigma \)-module if and only if, for each module \( L \), \( \text{Hom}(M, -) \) commutes with direct sums of the module \( L \).

**Proof.** The "only if" part is trivial. For the "if" part, begin with a family \( \{L_i|i \in I\} \) of modules; set \( L = \bigoplus\{L_i|i \in I\} \); and let \( \mu_i: L^{(I)} \to L \) denote the projection map. Now let \( f \in \text{Hom}(M, L) \) and define \( \bar{f}: M \to L \).
\[ L^{(i)}(\pi_{ij} f)(m) = y \in L, \text{ where } y = 0 \text{ if } i \neq j \text{ and } y = (\pi_j f)(m) \text{ if } i = j. \]

\[ f \text{ is a homomorphism and the assumption yields a finite subset } J \text{ of } I \text{ such that if } j \in I - J, \left(\mu_j f\right)(M) = 0 \in L. \text{ If } (\pi_j f)(M) \neq 0, \text{ then } \left(\pi_i \mu_j f\right)(M) \neq 0 \text{ and it follows that } i \in J. \] This shows that \( M \) is a \( \Sigma \)-module.

**Remark.** It can be shown that one need consider only countable direct sums of the various modules \( L \).

This theorem suggests the question: Is there one module \( T \) so that if \( \text{Hom}(M, -) \) commutes with direct sums of \( T \) then \( M \) is a \( \Sigma \)-module? Such a module \( T \) would serve as a "test module" for \( \Sigma \)-modules. In fact we adopt this as our definition of a test module. We will show next that test modules (always) exist and are quite familiar modules.

**Theorem 2.** A module \( T \) is a test module if and only if, for each module \( X \neq 0 \), \( \text{Hom}(X, T) \neq 0 \).

**Proof.** Suppose \( T \) is a test module and \( \text{Hom}(X, T) = 0 \) for a module \( X \). Then \( \text{Hom}(X^{(N)}, T) = 0 \) so \( X^{(N)} \) is a \( \Sigma \)-module. This is impossible if \( X \neq 0 \). Thus \( X = 0 \).

Conversely, suppose \( T \) is a module satisfying: For each module \( X \neq 0 \), \( \text{Hom}(X, T) \neq 0 \). Further assume that \( X \) is a module such that \( \text{Hom}(X, -) \) commutes with direct sums of \( T \). We must show that \( X \) is a \( \Sigma \)-module. Consider any module \( L \) and \( f \in \text{Hom}(X, L^{(N)}) \). Assume, by way of contradiction, that the set \( K = \{ n \in N \mid (p_n f)(X) \neq 0 \} \) is an infinite set, where \( p_n: L^{(N)} \to L \) is the \( n \)th projection. For each \( k \in K \), select \( \neq b_k \in \text{Hom}(p_k f(X), M) \).

If \( n \in N \) and \( n \notin K \) let \( b_n = 0: p_n f(X) \to M \). If \( k \in K \) there exists \( x_k \in X \) such that \( b_k(p_k f(x_k)) \neq 0 \). Now put \( h = \bigoplus_{n \in N} b_n: \bigoplus_{n \in N} p_n f(X) \to M^{(N)} \).

One easily checks that \( hf \in \text{Hom}(X, M^{(N)}) \). Showing \( \pi_k (hf) \neq 0 \) if \( k \in K \) will contradict the fact that \( \text{Hom}(X, -) \) commutes with direct sums of \( T \).

Let \( k \in K \),

\[ hf(x_k) = h(f(x_k)) = h(p_n f(x_k)) = \left(\bigoplus_{n \in N} b_n\right) (p_n f(x_k)) = (b_n (p_n f(x_k))). \]

From above the \( k \)th component is nonzero. Thus the \( k \)th projection of \( hf \) is nonzero. With the help of Theorem 1, this completes the proof.

**Corollary.** A cogenerator (for the category of left \( R \)-modules) is a test module.

This shows, in answer to the question above, that test modules (always) exist but it raises another question. When is a test module a cogenerator? Before giving the answer we require the following fact.

**Lemma.** For a module \( M \) there is a submodule \( H \) of \( M \) and a simple
module $S$ such that $M/H$ can be embedded in $\mathcal{I}(S)$, the injective hull of $S$.

**Proof.** Choose $K \subseteq L \subseteq M$ with $L/K$ simple. If $L/K \subseteq M/K$ is not essential, choose $H/K \subseteq M/K$ such that $H/K \cap L/K = 0$ and $H/K$ is maximal with respect to this property. Then $(L + H)/H$ is simple and essential in $M/H$.

The next theorem may be of independent interest.

**Theorem 3.** For a ring $R$ the following are equivalent:

(a) every test module is a cogenerator;

(b) for each simple module $S$, and each submodule $L \subseteq \mathcal{I}(S)$, $\mathcal{I}(S)/L$ contains an isomorphic copy of $\mathcal{I}(S)$.

**Proof.** Assume (b) holds. Let $C$ be a test module and consider a simple module $S \neq 0$. By Theorem 2 we choose $0 \neq f \in \text{Hom}(\mathcal{I}(S), C)$. By hypothesis $\mathcal{I}(S) \cong \mathcal{I}(S)/\text{Ker } f \subset C$ so $C$ is a cogenerator.

Now assume (b) fails. Then for some simple module $S$, we have $N \subseteq \mathcal{I}(S)$ such that $\mathcal{I}(S)/N$ does not contain a copy of $\mathcal{I}(S)$. Let

$$C = (\mathcal{I}(S)/N) \oplus \left( \bigoplus \{ \mathcal{I}(U) \mid U \text{ is simple and } U \not\cong S \} \right) \oplus \left( \bigoplus \{ M \mid M \subseteq \mathcal{I}(S) \} \right).$$

$C$ does not contain a copy of $\mathcal{I}(S)$ so is not a cogenerator. However, we will show that $C$ is a test module by using Theorem 2.

Let $X \neq 0$ be a module. By the Lemma we choose a simple module $U$ such that $X/Y \subseteq \mathcal{I}(U)$ for some submodule $Y \subseteq X$. If $U \cong S$ then, trivially, $\text{Hom}(X, C) \neq 0$. We consider the two cases (1) $X/Y \cong \mathcal{I}(S)$, (2) $X/Y \subset \mathcal{I}(S)$, but $X/Y \not\cong \mathcal{I}(S)$. In the first case, use $\mathcal{I}(S)/N$ to get the nonzero element of $\text{Hom}(X, C)$; and, in the second case, use one of the $M$'s, $M \subseteq \mathcal{I}(S)$. This completes the proof.

The authors would like to thank Professor E. Enochs for the clever construction in the proof of Theorem 3. We note that Tiwary [6] and Vamos [7] have shown that, over an integral domain $R$, $\mathcal{I}(S) \cong \mathcal{I}(S)/K$ for all simple modules $S$ and all submodules $K \subseteq \mathcal{I}(S)$, if and only if, $R_p$ is a PID for all prime ideals $P$ of $R$. Thus, for example, over a Dedekind domain a test module is a cogenerator.

The condition (b) of Theorem 3 appears to be interesting. Among the things it implies are: The socle of $\mathcal{I}(S)/K$, $K \subseteq \mathcal{I}(S)$, consists of copies of $S$ and is essential in $\mathcal{I}(S)/K$.

**REFERENCES**


