ABSTRACT. If $U/G$ represents a Riemann surface as the disk $U$ modulo a discontinuous group $G$ and if $L^p/G$ denotes the $L^p$ functions on the circle which are $G$ invariant, then it is shown that $L^p/G = N_p \oplus K_p$ if and only if $H^p/G$ and $H^q/G$ are naturally dual. Here $K_p$ is the subset of $L^p/G$ consisting of those functions which are invariant and whose conjugates are invariant; $N_p$ is $E(H^p) \cap E(H^q_0)$ where $E$ is the conditional expectation operator. $H^p$ is the space of boundary values of holomorphic functions and $1 < p < \infty$.

Let $G$ denote a group of linear fractional transformations acting on the disk $U = \{z: |z| < 1\}$ and let $L^p/G$ denote the subspaces of $G$-automorphic functions of $L^p$ of the unit circle, $1 < p < \infty$. If $K_p$ denotes the subspace of functions in $L^p/G$ whose conjugates also lie in $L^p/G$ and if $N_p = E(H^p) \cap E(H^q_0)$ where $E$ is the conditional expectation operator, then Forelli [3] has shown that $L^p/G = N_p \oplus K_p$ whenever $G$ is the cover group of a compact bordered surface.

The main result of this paper (Theorem 1) shows that such a decomposition holds for general $G$ if and only if the dual of $H^p/G$ is naturally isomorphic to $H^q/G$, and this is equivalent to $L^p/G = H^p/G \oplus \bar{H}^q/G$. Certain ancillary results concerning projections on $H^p$ and $H^\infty$ are considered. The results here are closely related to those in [1], [3], [4], and [5].

If $G$ is a group of automorphisms of the disk $U$, then there is associated to $G$ the sigma-field $\Sigma_G$ of $G$-invariant subsets of the sigma-field of measurable subsets of the circle. There is the conditional expectation operator $E$ used in [3] which associates to each integrable function an integrable $G$-invariant function. The linear operator $E$ induces by restriction a continuous projection $E_p$ on $L^p \to L^p$ for every $p, 1 \leq p$, and

$$\int_A f(w) \, dw = \int_A E(f)(w) \, dw$$

Received by the editors November 13, 1973 and, in revised form April 15, 1974. 

Copyright © 1975, American Mathematical Society
for each $A \in \Sigma_G$ and $E(f)$ is $G$-invariant, $E_p$ being a continuous projection $L^p = E_p(L^p) \oplus \text{Ker } E_p$ and $f \in \text{Ker } E_p$ if and only if $f \in L^p$ and $\int_A f(w) \, dw = 0$ for each $A \in \Sigma_G$. In addition, $E_p$ has these properties:

$$\|E_p(f)\|_p \leq \|f\|_p, \quad 1 \leq p < \infty,$$

$$E_p(kf) = kE_p(f) \quad \text{where } k \in E(L^q) \text{ and } f \in L^p.$$

The indices $p$ and $q$ will refer to conjugate indices throughout. The adjoint of $E_p$ is $E_p^*$. For more details concerning the operator $E$ in this setting, see [3] and [6].

For each subspace $A$ of $L^p$, $A/G$ denotes those members of $A$ which are $G$-invariant, while for each $x \in L^1$, $^*G$ denotes that conjugate of $x$ which has mean-value zero. $H^p$ denotes that subspace of $L^p$ which consists of the boundary values of holomorphic functions. The subscript zero denotes those members of $H^p$ which have mean value zero; the bar denotes complex conjugation. Unless otherwise noted, the $L^p$-spaces are considered as complex vector spaces of complex-valued functions. The decomposition of $L^p$ as $H^p \oplus \overline{H^p}$ for $1 < p < \infty$ corresponds to a projection $T_p$ where $T_p(L^p) = H^p$. The pairing $(x, y) = \int x(w)y(w) \, dw$ for $x \in L^p$ and $y \in L^q$ gives $H^p \perp$, the orthogonal complement of $H^p$, as $H^q_0$ and $\overline{H^p_0} = \overline{H^q}$.

This pairing will be used consistently. It exhibits the dual of $H^p$ (considered as holomorphic functions on the disk $U$) as the space $H^q$ of functions holomorphic on the surface dual to $U$ viz., the surface $\{Z: |Z| > 1\}$. For an arbitrary group $G$, $H^p/G$ and $H^q/G$ are not dual, though $L^p/G$ and $L^q/G$ are always so.

The fact that $T_p$ is a continuous projection for each $p$, $1 < p < \infty$, will be used, in particular, in the proof of Theorem 1. In what follows, $1 < p < \infty$.

**Lemma 1.** The space $H^p/G \oplus \overline{H^p_0}/G$ is a closed subspace of $L^p/G$ which coincides with the set of $x$ such that both $x$ and $x^*$ belong to $L^p/G$.

The proof of this lemma appears in [3, p. 372].

Denote by $F_g$ the $L^p$-closure of the complex linear space of the functions $E(|D_g|*)$ for $g \in G$ (see [3, p. 370]). Define also $N_p = \text{the } L^p$-closure of $E(H^p) \cap E(H^q_0)$.

**Lemma 2.** The orthogonal complement of $F_g$ in $L^q/G$ is $H^q/G \oplus \overline{H^q_0}/G$.

The proof of this lemma appears in [3, pp. 370–372].

**Lemma 3.** $F_g$ is a subset of $N_p$.

**Proof.** The function $f(w) = (w + b)/(w - b)$ belongs to $\overline{H^p}$ for every $p$.
when $|b| < 1$; and when $b = g(0), f = |Dg^{-1}| + i|Dg^{-1}|^*$. For $A \in \Sigma_G$, $g(A)$ and $A$ have the same measure for every $g \in G$. So $\int_B \, dw = \int_A |Dg^{-1}| \, dw = \int_A \, dw$, where $B = g^{-1}(A)$. Thus $E(|Dg^{-1}|) = 1$ for every $g \in G$. Now $E(|Dg^{-1}|) + iE(|Dg^{-1}|^*) = E(f) \in E(\overline{H}^0)$. So $iE(|Dg^{-1}|^*) = E(f) - 1 = E(f - 1) \in E(\overline{H}^0)$. Therefore, $E(|Dg^{-1}|^*) \in E(\overline{H}^0)$. Because $|Dg^{-1}|^*$ is real, $E(|Dg^{-1}|^*) \in E(\overline{H}^0)$. So $E(|Dg^{-1}|^*) \in E(H^p) \cap E(\overline{H}^0)$ for every $g \in G$ which suffices to prove the lemma.

**Lemma 4.** $F_p = N_p$ and the orthogonal complement of $N_p$ is $H^q/G \oplus \overline{H}^q_0/G$.

**Proof.** The result of the previous lemma implies that $N_p^\perp$ is a subset of $F_p^\perp$. So it suffices to prove that $H^q_0/G \oplus \overline{H}^q/G$ is a subset of $N_p^\perp$, for then the result of Lemma 2 implies equality. If $f \in E(H^p) \cap E(\overline{H}^0)$ and $k_1 + k_2 \in H^q_0/G \oplus \overline{H}^q/G$, then $\int f k_1 + \int f k_2 = \int k_1 E f_1 + \int k_2 E f_2$ for some $f_1 \in H^p$ and $f_2 \in \overline{H}^0$. So

$$\int (k_1 + k_2) f = \int E(f_1 k_1) + \int E(f_2 k_2) = \int f_1 k_1 + \int f_2 k_2,$$

and each of the last two integrals is zero since the orthogonal complement of $H^p$ is $H^q_0$ and that of $\overline{H}^0$ is $\overline{H}^q$. So

$$H^q_0/G \oplus \overline{H}^q/G \subset (E(H^p) \cap E(\overline{H}^0))^\perp = N_p^\perp.$$

Thus, $F_p^\perp = N_p^\perp$. But then $F_p$ is dense in $N_p$ and since $F_p$ is closed, equality holds.

**Lemma 5.** The orthogonal complement of $E(H^p)$ is $H^q_0/G$.

The proof of this lemma involves straightforward computation.

**Theorem 1.** Suppose that for some $p, 1 < p < \infty$,

$$L^p/G = (H^p/G \oplus \overline{H}^p_0/G) \oplus N_p.$$

Then there is a map $\alpha$ from the dual of $\overline{H}^q/G$ to $H^p/G$ which is an isomorphism and which satisfies

$$F(f) = \int f(w) \alpha(F)(w) \, dw$$

for every $f \in \overline{H}^q/G$ and every $F$ in the dual of $\overline{H}^q/G$.

Conversely, if there exists an isomorphism $\alpha$ from the dual of $\overline{H}^q/G$ to $H^p/G$ which satisfies (2), then the direct sum decomposition in (1) holds.

**Proof.** I first assume the isomorphism $\alpha$ exists and show that the direct
DUALITY BETWEEN $H^p$ AND $H^q$ AND ASSOCIATED PROJECTIONS

345

sum decomposition in (1) obtains. First, I need that $L^p/G = H^p/G \oplus \overline{H^q/G}$.

It is clear that $H^p/G$ and $\overline{H^q/G}$ are closed subspaces of $L^p/G$. Moreover, the intersection of these subspaces is zero. For suppose $k \in H^p/G$. Then there is a unique functional $F$ defined on $\overline{H^q/G}$ by $F(f) = \int f(w)k(w) \, dw$ for every $f \in \overline{H^q/G}$. But if $k \in H^q/G$, then $F$ is zero and since $\alpha$ is an isomorphism, $k$ is also zero. To conclude, I will show that $L^p/G \subset H^p/G + \overline{H^q/G}$.

Now $x \in L^p/G$ implies that $x$ determines a functional $F_x$ on $\overline{H^q/G}$ by the rule

$$F_x(f) = \int x(w)f(w) \, dw, \quad f \in \overline{H^q/G}.$$ 

So there is an $\alpha(F_x) \in H^p/G$ such that $\int f(w)x(w) \, dw = \int f(w)\alpha(F_x)(w) \, dw$, for every $f \in \overline{H^q/G}$. Then $x - \alpha(F_x) \in \overline{H^q/G}$ and $x = \alpha(F_x) + y$ where $y \in H^q/G$ and $\alpha(F_x) \in H^p/G$. Thus, $L^p/G \subset H^p/G + \overline{H^q/G}$ and, therefore, $L^p/G = H^p/G \oplus \overline{H^q/G}$.

Now I will show that the decomposition in (1) holds. From the above there are two continuous projections, $p_1$ and $p_2$, on $L^p/G$ to $L^p/G$; the image of $p_1$ is $H^p/G$ and that of $p_2$ is $\overline{H^q/G}$. The sum $p_1 + p_2$ is a projection if and only if $p_1p_2 = p_2p_1 = 0$ [2, p. 514]. But this is clear since $H^p/G \cap \overline{H^q/G} = \{0\}$. Thus, $p_1 + p_2$ is a continuous projection whose image is $H^p/G \oplus \overline{H^p/G}$ and whose null space is $H^q/G \oplus \overline{H^q/G} = (H^q/G \oplus \overline{H^q/G}) \oplus N_p$. Thus, the first part of the theorem is proved.

Now I assume the decomposition in (1) and prove the existence of an isomorphism satisfying (2). The proof proceeds as follows. The assumption $L^p/G = N_p \oplus N^\perp_q$ leads to the conclusion $L^p/G = H^p/G \oplus \overline{H^q/G}$ and this decomposition leads directly to the desired result. In fact $L^p/G = H^p/G \oplus \overline{H^q/G}$ and the results of [7, pp. 110–111] give the required isomorphism. Thus, it suffices to show that $L^p/G = N_p \oplus N^\perp_q$ yields $L^p/G = H^p/G \oplus \overline{H^q/G}$.

The hypothesis $L^p/G = N_p \oplus (H^p/G \oplus \overline{H^0/G})$ gives, by duality, the same relation with $q$ replacing $p$. Therefore, I may assume that $1 < q \leq 2 < p < \infty$ and, as a consequence, $H^p/G \subset H^q/G$. I wish to assert that $\overline{H^q/G} = \overline{H^0/G} \oplus N_p$. For then $L^p/G = H^p/G \oplus \overline{H^q/G}$. Lemma 5 implies that $\overline{H^0/G} \oplus N_p \subset \overline{H^q/G}$. For the opposite inclusion suppose $k \in L^p/G$ and that $\int k(w)x(w) \, dw = 0$ for every $x \in \overline{H^q/G}$. Then $k = k_1 + k_2$ where $k_1 \in H^p/G$ and $k_2 \in \overline{H^0/G} + N_p$. So $\int k_1(w)x(w) \, dw = 0$ for every such $x$. But by the assumption $H^p/G \subset H^q/G$, one choice for $x$ is $k_1$. Thus $\int |k_1|^2 \, dw = 0$ so $k_1 = 0$. Therefore the sum $\overline{H^p/G} \oplus N_p = \overline{H^q/G}$ and the desired result follows.

In connection with the above result, the worst possibility occurs when,
say, $H^p(U/G)$ is not trivial but $H^q(U/G)$ is [5, p. 35]. Then $N_q$ is zero so $N_q \perp$ is $L^p/G$ and $N_p \subset N_{q} \perp$.

The following corollary gives the existence of two related projections.

**Corollary 1.** If, for some $p$, $1 < p < \infty$, $L^p/G = N_q^1 \oplus N_p$, then $E(H^p) = H^p/G \oplus N_p$ and $H^p = H^p/G \oplus C_p$ where $C_p = (N_p \oplus B_p) \cap H^p$; $B_p$ is the kernel of $E_p$.

**Proof.** That $E(H^p) = H^p/G \oplus N_p$ was obtained in the course of the proof of Theorem 1. Now $L^p = (H^p/G \oplus \overline{H}^p_0/G) \oplus N_p$. Therefore, if $x \in H^p$,

$$x = x_1 + x_2 + x_3, \quad x_1 \in H^p/G, \quad x_2 \in \overline{H}^p_0/G, \quad x_3 \in N_p + B_p.$$

Since $E(x) = x_1 + x_2 + E(x_3)$ and $E(H^p) = H^p/G \oplus N_p$ it follows that $x_2 = 0$. So $H^p \subset H^p/G + C_p$. It follows just as easily that $C_p$ is closed and $C_p \cap H^p/G = \{0\}$.

**Corollary 2.** If, for some $p$, $1 < p < \infty$, $L^p/G = N^1_p \oplus N^1_q$, then such a decomposition holds for every $r$ on the closed interval whose endpoints are $p$ and $q$ ($q$ is the index conjugate to $p$).

This corollary is a direct consequence of Theorem 1 and the Riesz convexity theorem [2, p. 525].

The next proposition gives a sufficient condition for the factorization in (1) to hold.

**Proposition 1.** If, for some $p$, $1 < p < \infty$, dim $N^1_p < \infty$, then $L^p/G = N^1_q \oplus N^1_p$ for every $p$, $1 < p < \infty$.

**Proof.** The definition of $N_p^1$ as the $L^p$-closure of the complex linear span of the functions $E(|Dg|^*)$ shows that all $N^1_r$ coincide in this case. So $N = N_p = N_q$ for every $r$, $1 < r < \infty$. Now $N_2$ and $H^2/G \oplus \overline{H}^2_0/G$, being orthogonal complements in $E(L^2)$, meet in zero only. If $p \geq 2$ and $x \in N_q^1$, then $x \in N_2$ so $N_q^1 \cap N = \{0\}$ when $p \geq 2$. If $L^r/G = (H^r/G \oplus \overline{H}^r_0/G) \oplus N_r$ holds for $r \geq 2$, then by duality it holds for all $r$, $1 < r < \infty$. Thus it suffices to prove such a direct sum decomposition when $r \geq 2$. For $r \geq 2$, $H^r/G \oplus \overline{H}^r_0/G$ and $N$ are closed and intersect in zero only. If $x \in L^r/G$, then $x \in L^2/G$ so

$$x = x_1 + x_2, \quad x_2 \in N, \quad x_1 \in H^r/G \oplus \overline{H}^r_0/G.$$

But then $x_1 = x - x_2 \in L^r/G$ and $x_1$ and its conjugate both belong to $L^r$ by Riesz's theorem. Moreover, the conjugate of $x_1$ is invariant so $x_1 \in H^r/G \oplus \overline{H}^r_0/G$. Thus $L^r/G \subset (H^r/G \oplus \overline{H}^r_0/G) \oplus N$, and the theorem is proved.
If the first homology group of $U/G$ is finitely generated, then $\dim N_p < \infty$ for every $p$. The proof of this appears in [3, pp. 372–379].

**Theorem 2.** If, for a given $p$, $1 < p < \infty$, the decomposition in (1) holds and $N_p \subset L^\infty$, then there is a continuous projection for $H^\infty$ to $H^\infty$ whose image is $H^\infty/G$.

**Proof.** The conclusion of Corollary 1 gives $H^p = H^p/G \oplus C_p$ and $E(H^p) = H^p/G \oplus N_p$. Moreover, if $x \in H^p$ and $x$ is a sum $x_1 + x_2$, then $E(x) = x_1 + E(x_2)$ so $x$ and $E(x)$ project to the same member of $H^p/G$. For $x \in H^\infty$ define $P(x)$ to be the $H^p/G$ component of $x$ when $x$ is viewed as a member of $H^p$. Now $E(L^\infty) \subset L^\infty$ so if $x \in H^\infty$, $E(x) \in L^\infty$. Thus, $x = P(x) + y$ and $E(x) = P(x) + E(y)$. The hypothesis $N_p \subset L^\infty$ implies that $E(y) \subset L^\infty$ so $E(x) \in L^\infty$. Thus, $P(x) \in H^p/G \cap L^\infty$ so $P(x) \in H^\infty/G$. Therefore, $P$ maps $H^\infty$ onto $H^\infty/G$. It is clear that $P$ is idempotent so it suffices to prove that it is a continuous map from $H^\infty$ to $H^\infty$. I show that the graph is closed. Suppose $x_n \to x$ and $P(x_n) \to y$ in $H^\infty$. Now in $H^p$

\[
x_n = P(x_n) + y_n, \quad E(x_n) = P(x_n) + E(y_n) \quad \text{and} \quad \|x_n - x\|_p \leq \|x_n - x\|_\infty.
\]

Therefore, $x_n$ converges to $x$ in $L^p$, and $E$ being continuous, $E(x_n)$ converges to $E(x)$ in $E(H^p)$. So $E(y_n)$ has a limit in $L^p$, say $z$. Also, $E(x) = y + z$ where $z \in N_p$ and $y \in H^\infty/G$. Because $P$ is continuous on $L^p$, $x = P(x) + \lim y_n$ and $\lim y_n \in C_p$. Since $E(x) = y + z$ and $x$ and $E(x)$ have the same projection, $P(x) = y$ and $P$ is continuous on $H^\infty$. This completes the proof.

When $U/G$ represents a compact bordered surface, $N_p \subset L^\infty/G$ for $1 < p < \infty$. So the factorizations (1) hold and the continuous projection from $H^\infty$ to $H^\infty/G$ also exists.

On considering the representation of a Riemann surface as the disk modulo a discontinuous group $G$ it would appear that a basic question concerning $H^p$ spaces is: When is a given closed subspace of $H^p$ itself the $H^p$ space of some surface? A problem somewhat similar to this for $L^p$ has been treated by Andô [6] using the expectation operator $E$.

**REFERENCES**


DEPARTMENT OF MATHEMATICS, DEPAUL UNIVERSITY, CHICAGO, ILLINOIS 60614