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DUALITY BETWEEN H^p AND H^q AND ASSOCIATED PROJECTIONS

WALTER PRANGER

ABSTRACT. If U/G represents a Riemann surface as the disk U modulo a discontinuous group G and if L^p/G denotes the L^p functions on the circle which are G invariant, then it is shown that $L^p/G = N_p \oplus K_p$ if and only if H^p/G and $\bar{H}^q/G$ are naturally dual. Here K_p is the subset of L^p/G consisting of those functions which are invariant and whose conjugates are invariant; N_p is $E(H^p) \cap E(\bar{H}_0^p)$ where E is the conditional expectation operator. H^p is the space of boundary values of holomorphic functions and $1 < p < \infty$.

Let G denote a group of linear fractional transformations acting on the disk $U = \{Z: |Z| < 1\}$ and let L^p/G denote the subspaces of G-automorphic functions of L^p of the unit circle, $1 < p < \infty$. If K_p denotes the subspace of functions in L^p/G whose conjugates also lie in L^p/G and if $N_p = E(H^p) \cap E(\bar{H}_0^p)$ where E is the canditional expectation operator, then Forelli [3] has shown that $L^p/G = N_p \oplus K_p$ whenever G is the cover group of a compact bordered surface.

The main result of this paper (Theorem 1) shows that such a decomposition holds for general G if and only if the dual of H^p/G is naturally isomorphic to $\bar{H}^q/G$, and this is equivalent to $L^p/G = H^p/G \oplus \bar{H}^q/G^{\perp}$. Certain ancillary results concerning projections on H^p and H^∞ are considered. The results here are closely related to those in [1], [3], [4], and [5].

If G is a group of automorphisms of the disk U, then there is associated to G the sigma-field Σ_G of G-invariant subsets of the sigma-field of measurable subsets of the circle. There is the conditional expectation operator E used in [3] which associates to each integrable function an integrable G-invariant function. The linear operator E induces by restriction a continuous projection E_p on $L^p \to L^p$ for every p, $1 \leq p$, and

$$\int_A f(w)\,dw = \int_A E(f)(w)\,dw$$

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for each $A \in \Sigma_G$ and $E(f)$ is G-invariant, E_p being a continuous projection $L^p = E_p(L^p) \oplus \operatorname{Ker} E_p$ and $f \in \operatorname{Ker} E_p$ if and only if $f \in L^p$ and $\int_A f(w)\,dw = 0$ for each $A \in \Sigma_G$. In addition, E_p has these properties:

$$\|E_p(f)\|_p \leq \|f\|_p, \qquad 1 \leq p < \infty,$$

$$E_1(kf) = kE(f) \quad \text{where } k \in E(L^q) \text{ and } f \in L^p.$$

The indices p and q will refer to conjugate indices throughout. The adjoint of E_p is E_q. For more details concerning the operator E in this setting, see [3] and [6].

For each subspace A of L^p, A/G denotes those members of A which are G-invariant, while for each $x \in L^1$, * denotes that conjugate of x which has mean-value zero. H^p denotes that subspace of L^p which consists of the boundary values of holomorphic functions. The subscript zero denotes those members of H^p which have mean value zero; the bar denotes complex conjugation. Unless otherwise noted, the L^p-spaces are considered as complex vector spaces of complex-valued functions. The decomposition of L^p as $H^p \oplus \bar{H}_0^p$ for $1 < p < \infty$ corresponds to a projection T_p where $T_p(L^p) = H^p$. The pairing $\langle x, y\rangle = \int x(w)y(w)\,dw$ for $x \in L^p$ and $y \in L^q$ gives $H^{p\perp}$, the orthogonal complement of H^p, as H_0^q and $\bar{H}_0^{p\perp} = \bar{H}^{q\perp}$. This pairing will be used consistently. It exhibits the dual of H^p (considered as holomorphic functions on the disk U) as the space H^q of functions holomorphic on the surface dual to U viz., the surface $\{Z: |Z| > 1\}$. For an arbitrary group G, H^p/G and H^q/G are not dual, though L^p/G and L^q/G are always so.

The fact that T_p is a continuous projection for each p, $1 < p < \infty$, will be used, in particular, in the proof of Theorem 1. In what follows, $1 < p < \infty$.

Lemma 1. *The space $H^p/G \oplus \bar{H}_0^p/G$ is a closed subspace of L^p/G which coincides with the set of x such that both x and x^* belong to L^p/G.*

The proof of this lemma appears in [3, p. 372].

Denote by F_p the L^p-closure of the complex linear space of the functions $E(|Dg|^*)$ for $g \in G$ (see [3, p. 370]). Define also N_p = the L^p-closure of $E(H^p) \cap E(\bar{H}_0^p)$.

Lemma 2. *The orthogonal complement of F_p in L^q/G is $H^q/G \oplus \bar{H}_0^q/G$.*

The proof of this lemma appears in [3, pp. 370–372].

Lemma 3. *F_p is a subset of N_p.*

Proof. The function $f(w) = (w + b)/(w - b)$ belongs to $\bar{H}^p$ for every p

when $|b| < 1$; and when $b = g(0)$, $f = |Dg^{-1}| + i|Dg^{-1}|^*$. For $A \in \Sigma_G$, $g(A)$ and A have the same measure for every $g \in G$. So $\int_B dw = \int_A |Dg^{-1}|\, dw = \int_A dw$, where $B = g^{-1}(A)$. Thus $E(|Dg^{-1}|) = 1$ for every $g \in G$. Now $E(|Dg^{-1}|) + iE(|Dg^{-1}|^*) = E(f) \in E(\bar{H}^p)$. So $iE(|Dg^{-1}|^*) = E(f) - 1 = E(f-1) \in E(\bar{H}_0^p)$. Therefore, $E(|Dg^{-1}|^*) \in E(\bar{H}_0^p)$. Because $|Dg^{-1}|^*$ is real, $E(|Dg^{-1}|^*) \in E(H_0^p)$. So $E(|Dg^{-1}|^*) \in E(H^p) \cap E(\bar{H}_0^p)$ for every $g \in G$ which suffices to prove the lemma.

Lemma 4. *$F_p = N_p$ and the orthogonal complement of N_p is $H^q/G \oplus \bar{H}_0^q/G$.*

Proof. The result of the previous lemma implies that $N_p^\perp$ is a subset of $F_p^\perp$. So it sufficies to prove that $H_0^q/G \oplus \bar{H}^q/G$ is a subset of $N_p^\perp$, for then the result of Lemma 2 implies equality. If $f \in E(H^p) \cap E(\bar{H}_0^p)$ and $k_1 + k_2 \in H_0^q/G \oplus \bar{H}^q/G$, then $\int fk_1 + \int fk_2 = \int k_1 E f_1 + \int k_2 E f_2$ for some $f_1 \in H^p$ and $f_2 \in \bar{H}_0^p$. So

$$\int (k_1 + k_2) f = \int E(f_1 k_1) + \int E(f_2 k_2) = \int f_1 k_1 + \int f_2 k_2,$$

and each of the last two integrals is zero since the orthogonal complement of H^p is H_0^q and that of $\bar{H}_0^p$ is $\bar{H}^q$. So

$$H_0^q/G \oplus \bar{H}^q/G \subset (E(H^p) \cap E(\bar{H}_0^p))^\perp = N_p^\perp.$$

Thus, $F_p^\perp = N_p^\perp$. But then F_p is dense in N_p and since F_p is closed, equality holds.

Lemma 5. *The orthogonal complement of $E(H^p)$ is H_0^q/G.*

The proof of this lemma involves straightforward computation.

Theorem 1. *Suppose that for some p, $1 < p < \infty$,*

$$L^p/G = (H^p/G \oplus \bar{H}_0^p/G) \oplus N_p. \tag{1}$$

Then there is a map α from the dual of $\bar{H}^q/G$ to H^p/G which is an isomorphism and which satisfies

$$F(f) = \int f(w)\alpha(F)(w)\, dw \tag{2}$$

for every $f \in \bar{H}^q/G$ and every F in the dual of $\bar{H}^q/G$.

Conversely, if there exists an isomorphism α from the dual of $\bar{H}^q/G$ to H^p/G which satisfies (2), *then the direct sum decomposition in* (1) *holds.*

Proof. I first assume the isomorphism α exists and show that the direct

sum decomposition in (1) obtains. First, I need that $L^p/G = H^p/G \oplus \bar{H}^q/G^\perp$. It is clear that H^p/G and $\bar{H}^q/G^\perp$ are closed subspaces of L^p/G. Moreover, the intersection of these subspaces is zero. For suppose $k \in H^p/G$. Then there is a unique functional F defined on $\bar{H}^q/G$ by $F(f) = \int f(w)k(w)\,dw$ for every $f \in \bar{H}^q/G$. But if $k \in \bar{H}^q/G^\perp$, then F is zero and since α is an isomorphism, k is also zero. To conclude, I will show that $L^p/G \subset H^p/G + \bar{H}^q/G^\perp$. Now $x \in L^p/G$ implies that x determines a functional F_x on $\bar{H}^q/G$ by the rule

$$F_x(f) = \int x(w)f(w)\,dw, \quad f \in \bar{H}^q/G.$$

So there is an $\alpha(F_x) \in H^p/G$ such that $\int f(w)x(w)\,dw = \int f(w)\alpha(F_x)(w)\,dw$, for every $f \in \bar{H}^q/G$. Then $x - \alpha(F_x) \in \bar{H}^q/G^\perp$ so $x = \alpha(F_x) + y$ where $y \in \bar{H}^q/G^\perp$ and $\alpha(F_x) \in H^p/G$. Thus, $L^p/G \subset H^p/G + \bar{H}^q/G^\perp$ and, therefore, $L^p/G = H^p/G \oplus \bar{H}^q/G^\perp$.

Now I will show that the decomposition in (1) holds. From the above there are two continuous projections, p_1 and p_2, on L^p/G to L^p/G; the image of p_1 is H^p/G and that of p_2 is $\bar{H}_0^p/G$. The sum $p_1 + p_2$ is a projection if and only if $p_1p_2 = p_2p_1 = 0$ [2, p. 514]. But this is clear since $H^p/G \cap \bar{H}_0^p/G = \{0\}$. Thus, $p_1 + p_2$ is a continuous projection whose image is $H^p/G \oplus \bar{H}_0^p/G$ and whose null space is $H^q/G^\perp \cap \bar{H}_0^q/G^\perp = (H^q/G \oplus \bar{H}_0^q/G)^\perp = N_p$. Thus, the first part of the theorem is proved.

Now I assume the decomposition in (1) and prove the existence of an isomorphism satisfying (2). The proof proceeds as follows. The assumption $L^p/G = N_p \oplus N_q^\perp$ leads to the conclusion $L^p/G = H^p/G \oplus \bar{H}^q/G^\perp$ and this decomposition leads directly to the desired result. In fact $L^p/G = H^p/G \oplus \bar{H}^q/G^\perp$ and the results of [7, pp. 110–111] give the required isomorphism. Thus, it suffices to show that $L^p/G = N_p \oplus N_q^\perp$ yields $L^p/G = H^p/G \oplus \bar{H}^q/G^\perp$.

The hypothesis $L^p/G = N_p \oplus (H^p/G \oplus \bar{H}_0^p/G)$ gives, by duality, the same relation with q replacing p. Therefore, I may assume that $1 < q \leq 2 \leq p < \infty$ and, as a consequence, $H^p/G \subset H^q/G$. I wish to assert that $\bar{H}^q/G^\perp = \bar{H}_0^p/G \oplus N_p$. For then $L^p/G = H^p/G \oplus \bar{H}^q/G^\perp$. Lemma 5 implies that $\bar{H}_0^p/G \oplus N_p \subset \bar{H}^q/G^\perp$. For the opposite inclusion suppose $k \in L^p/G$ and that $\int k(w)x(w)\,dw = 0$ for every $x \in \bar{H}^q/G$. Then $k = k_1 + k_2$ where $k_1 \in H^p/G$ and $k_2 \in \bar{H}_0^p/G + N_p$. So $\int k_1(w)x(w)\,dw = 0$ for every such x. But by the assumption $H^p/G \subset H^q/G$, one choice for x is $\bar{k}_1$. Thus $\int |k_1|^2\,dw = 0$ so $k_1 = 0$. Therefore the sum $\bar{H}^p/G \oplus N_p = \bar{H}^q/G^\perp$ and the desired result follows.

In connection with the above result, the worst possibility occurs when,

say, $H^p(U/G)$ is not trivial but $H^q(U/G)$ is [5, p. 35]. Then N_q is zero so $N_q^\perp$ is L^p/G and $N_p \subset N_q^\perp$.

The following corollary gives the existence of two related projections.

Corollary 1. *If, for some* p, $1 < p < \infty$, $L^p/G = N_q^\perp \oplus N_p$, *then* $E(H^p) = H^p/G \oplus N_p$ *and* $H^p = H^p/G \oplus C_p$ *where* $C_p = (N_p \oplus B_p) \cap H^p$; B_p *is the kernel of* E_p.

Proof. That $E(H^p) = H^p/G \oplus N_p$ was obtained in the course of the proof of Theorem 1. Now $L^p = ((H^p/G \oplus \bar{H}_0^p/G) \oplus N_p) \oplus B_p$. Therefore, if $x \in H^p$,

$$x = x_1 + x_2 + x_3, \qquad x_1 \in H^p/G, \quad x_2 \in \bar{H}_0^p/G, \quad x_3 \in N_p + B_p.$$

Since $E(x) = x_1 + x_2 + E(x_3)$ and $E(H^p) = H^p/G \oplus N_p$ it follows that $x_2 = 0$. So $H^p \subset H^p/G + C_p$. It follows just as easily that C_p is closed and $C_p \cap H^p/G = \{0\}$.

Corollary 2. *If, for some* p, $1 < p < \infty$, $L^p/G = N_p \oplus N_q^\perp$, *then such a decomposition holds for every* r *on the closed interval whose endpoints are* p *and* q (q *is the index conjugate to* p).

This corollary is a direct consequence of Theorem 1 and the Riesz convexity theorem [2, p. 525].

The next proposition gives a sufficient condition for the factorization in (1) to hold.

Proposition 1. *If, for some* p, $1 < p < \infty$, $\dim N_p < \infty$, *then* $L^p/G = N_q^\perp \oplus N_p$ *for every* p, $1 < p < \infty$.

Proof. The definition of N_p as the L^p-closure of the complex linear span of the functions $E(|Dg|^*)$ shows that all N_r coincide in this case. So $N = N_p = N_r$ for every r, $1 < r < \infty$. Now N_2 and $H^2/G \oplus \bar{H}_0^2/G$, being orthogonal complements in $E(L^2)$, meet in zero only. If $p \geq 2$ and $x \in N_q^\perp$, then $x \in N_2$ so $N_q^\perp \cap N = \{0\}$ when $p \geq 2$. If $L^r/G = (H^r/G \oplus \bar{H}_0^r/G) \oplus N_r$ holds for $r \geq 2$, then by duality it holds for all r, $1 < r < \infty$. Thus it suffices to prove such a direct sum decomposition when $r \geq 2$. For $r \geq 2$, $H^r/G \oplus \bar{H}_0^r/G$ and N are closed and intersect in zero only. If $x \in L^r/G$, then $x \in L^2/G$ so

$$x = x_1 + x_2, \qquad x_2 \in N, \quad x_1 \in H^r/G \oplus \bar{H}_0^r/G.$$

But then $x_1 = x - x_2 \in L^r/G$ and x_1 and its conjugate both belong to L^r by Riesz's theorem. Moreover, the conjugate of x_1 is invariant so $x_1 \in H^r/G \oplus \bar{H}_0^r/G$. Thus $L^r/G \subset (H^r/G \oplus \bar{H}_0^r/G) \oplus N$, and the theorem is proved.

If the first homology group of U/G is finitely generated, then $\dim N_p < \infty$ for every p. The proof of this appears in [3, pp. 372–379].

Theorem 2. *If, for a given p, $1 < p < \infty$, the decomposition in* (1) *holds and $N_p \subset L^\infty$, then there is a continuous projection for H^∞ to H^∞ whose image is H^∞/G.*

Proof. The conclusion of Corollary 1 gives $H^p = H^p/G \oplus C_p$ and $E(H^p) = H^p/G \oplus N_p$. Moreover, if $x \in H^p$ and x is a sum $x_1 + x_2$, then $E(x) = x_1 + E(x_2)$ so x and $E(x)$ project to the same member of H^p/G. For $x \in H^\infty$ define $P(x)$ to be the H^p/G component of x when x is viewed as a member of H^p. Now $E(L^\infty) \subset L^\infty$ so if $x \in H^\infty$, $E(x) \in L^\infty$. Thus, $x = P(x) + y$ and $E(x) = P(x) + E(y)$. The hypothesis $N_p \subset L^\infty$ implies that $E(y) \subset L^\infty$ so $E(x) \in L^\infty$. Thus, $P(x) \in H^p/G \cap L^\infty$ so $P(x) \in H^\infty/G$. Therefore, P maps H^∞ onto H^∞/G. It is clear that P is idempotent so it suffices to prove that it is a continuous map from H^∞ to H^∞. I show that the graph is closed. Suppose $x_n \to x$ and $P(x_n) \to y$ in H^∞. Now in H^p

$$x_n = P(x_n) + y_n, \qquad E(x_n) = P(x_n) + E(y_n) \quad \text{and} \quad \|x_n - x\|_p \leq \|x_n - x\|_\infty.$$

Therefore, x_n converges to x in L^p, and E being continuous, $E(x_n)$ converges to $E(x)$ in $E(H^p)$. So $E(y_n)$ has a limit in L^p, say z. Also, $E(x) = y + z$ where $z \in N_p$ and $y \in H^\infty/G$. Because P is continuous on L^p, $x = P(x) + \lim y_n$ and $\lim y_n \in C_p$. Since $E(x) = y + z$ and x and $E(x)$ have the same projection, $P(x) = y$ and P is continuous on H^∞. This completes the proof.

When U/G represents a compact bordered surface, $N_p \subset L^\infty/G$ for $1 < p < \infty$. So the factorizations (1) hold and the continuous projection from H^∞ to H^∞/G also exists.

On considering the representation of a Riemann surface as the disk modulo a discontinuous group G it would appear that a basic question concerning H^p spaces is: When is a given closed subspace of H^p itself the H^p space of some surface? A problem somewhat similar to this for L^p has been treated by Andô [6] using the expectation operator E.

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DEPARTMENT OF MATHEMATICS, DEPAUL UNIVERSITY, CHICAGO, ILLINOIS 60614