PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 49, Number 2, June 1975

SEMILATTICES ON PEANO CONTINUA

W. WILEY WILLIAMS

ABSTRACT. A continuum is cell-cyclic if every cyclic element is an n-cell for some integer n. It is shown that every cell-cyclic Peano continuum admits a topological semilattice.

By a semigroup we mean a Hausdorff topological space together with an associative multiplication. One of the oldest problems in semigroup theory is: "Given a space X with topological properties P, does X admit the structure of a semigroup having algebraic properties Q?". In the case when Q is "commutative and idempotent", X is said to be a semilattice, and another approach is open. If one can define on X a partial order so that the operation $\wedge: X \times X \to X$ defined by $\bigwedge(a, b) = \text{l.u.b.}\{a, b\}$ is continuous, then $(X, \wedge)$ is a semilattice. Knight has shown [3] that any Peano (locally connected metric) continuum admits a partial order with closed graph.

In order for a Peano continuum to be a semilattice it must be acyclic [6], however not all acyclic Peano continua admit a semilattice structure. Lawson and the author have shown [5] that any semilattice on a finite-dimensional Peano continuum which is not one-dimensional contains a two-cell. Thus the example given by Borsuk in [1] is a two-dimensional Peano continuum which does not admit a semilattice. We prove here that a Peano continuum every cyclic element of which is an n-cell for some integer n admits a semilattice.

We shall use the cyclic element notation and results of Whyburn [7] and Kuratowski and Whyburn [4], slightly modified in the following way. In a Peano continuum X, we say x separates a and b if each arc from a to b contains x, and a cyclic element D separates a and b if each arc from a to b meets D. $E(a, b)$ denotes the set of points which separate a and b (including a and b) and is a compact partially ordered set under the ordering $x \leq y$ iff x separates y and a. $C(a, b)$ denotes the cyclic chain from

Received by the editors December 20, 1973 and, in revised form, April 12, 1974.
AMS (MOS) subject classifications (1970). Primary 22A15; Secondary 06A20.
Key words and phrases. Peano continuum, cyclic element, cell-cyclic, topological semilattice, retract.

Copyright © 1975, American Mathematical Society

a to b and is $\{x$: some arc from a to b contains $x\}$. Given a point a and a cyclic chain $C(p, q)$, if $a \notin C(p, q)$ there is a unique element x of $C(p, q)$ such that x separates each element of $C(p, q)$ from a. Denote x by $P(C(p, q), a)$. If $a \in C(p, q)$, set $P(C(p, q), a) = a$. We will use I to denote the unit interval under min multiplication.

Lemma 1 ([4, p. 70]). *Let X be a Peano continuum, and let C be a fixed cyclic chain of X. The function $f_C\colon X \to C$ defined by $f_C(x) = P(C, x)$ is a monotone retraction mapping $X - C$ into the boundary of C.*

Note that since a cyclic element is a cyclic chain between any pair of its points, the above holds for cyclic elements as well.

Lemma 2. *Let a be an element of a cyclic chain C in a Peano continuum X, and let $\epsilon > 0$. There exists $\delta > 0$ such that for each x in $X - C$, $P(C, x)$ in $B(a, \delta) - \{a\}$ (the deleted open ball about a) implies x is in $B(a, \epsilon)$, and also x in $B(a, \delta)$ implies $P(C, x)$ is in $B(a, \epsilon)$.*

Proof. If a is in the interior of C, choose $\delta < \epsilon$ such that $B(a, \delta) \subset C^{\circ}$. Otherwise the components of $X - C$ form a null sequence at most [7], so there are a finite number of diameter $> \epsilon/2$. Choose a point x_i in each, and let

$$\delta < \min\left[\{d(a, P(C, x_i)) \mid d(a, P(C, x_i)) > 0\} \cup \{\epsilon/2\}\right].$$

Then if $P(C, y)$ is in $B(a, \delta) - \{a\}$, either $y \in C \cap B(a, \delta) \subset B(a, \epsilon)$ or y is in a component of $X - C$ of diameter $< \epsilon/2$. Thus

$$d(y, a) \leq d(y, P(C, y)) + d(P(C, y), a) < \epsilon/2 + \epsilon/2,$$

so the first implication holds. The second follows from Lemma 1.

The semilattice structure on X. A Peano continuum X is *cell-cyclic* iff every true cyclic element of X is an n-cell for some integer n. Let K_n be I^n under coordinatewise multiplication and K_n' be T^n under coordinatewise multiplication, where T is the subsemilattice $I \times \{0\} \cup \{0\} \times I$ of I^2. Then K_n and K_n' are semilattices on the n-cell, with the former having its minimum element on the boundary and the latter having its in the interior. Fix an element 0 of X.

For each cyclic element D of X define a function h_D as follows: If $P(D, 0)$ is on the boundary (interior) of D let $h_D\colon K_n\ (K_n') \to D$ be a homeomorphism mapping the minimum element of K_n (K_n') to $P(D, 0)$.

Lemma 3. *Let X be a cell-cyclic Peano continuum with a, b, and c in X. Suppose no cyclic element containing one of b and c separates the other from a. Then there exists a (possibly degenerate) cyclic element which separates any two of a, b, and c.*

Proof. *Case* I. Some cyclic element containing one element separates the other two. Then this cyclic element separates any two of a, b, and c.

Case II. No cyclic element containing one of a, b, and c separates the other two. This implies no two of a, b, and c lie in the same cyclic element. Then $E(a, c) \cap E(a, b)$ is the intersection of two compact totally ordered sets whose orderings agree where possible, i.e. on the intersection, and hence has a largest element d. If d separates b and c, then d is the required cyclic element. If d does not, then $C(d, b) \cap C(d, c)$ is a true cyclic element D of X separating each of b and c from a. Moreover the maximality of d in $E(a, c) \cap E(a, b)$ implies D separates b and c. This completes the proof.

Notation. Since the intersection of cyclic elements is a cyclic element, there is a smallest cyclic element which separates 0, a, and b. Denote it by $D(a, b)$.

Lemma 4. *Given three points a, b, and c in X, and any $x \in D(a, b)$ and $y \in D(b, c)$, $D(x, c) = D(a, y)$.*

Proof. It suffices to note that both $D(x, c)$ and $D(a, y)$ are the maximum cyclic element in $C(0, a) \cap C(0, b) \cap C(0, c)$.

To define the semilattice operation on X, we denote by $\wedge$ the operation on whichever of K_n or K'_n fits the context.

Given a and b in X, define

$$ab = h_{D(a,b)}(h^{-1}_{D(a,b)}(P(D(a, b), a)) \wedge h^{-1}_{D(a,b)}(P(D(a, b), b))).$$

Main Theorem. *Let X be a cell-cyclic Peano continuum. Then under the above operation X is a semilattice.*

Proof. The operation is obviously idempotent and commutative. It follows from Lemma 4 that it is associative. The proof of continuity will be by cases. First note that the operation is continuous when restricted to any cyclic element. Fix an open connected set U containing ab. We seek open sets V and W such that $a \in V$, $b \in W$ and $VW \subset U$.

Case I. a, b, and ab all distinct. If $D(a, b) = \{ab\}$ then the components of a and b in $X - \{ab\}$ are the required V and W. If $D(a, b)$ is a true cyclic element it is iseomorphic to K_n or K_n'. Thus there exist disjoint relatively open sets V' and W' of $D(a, b)$ containing $P(D(a, b), a)$ and $P(D(a, b), b)$, respectively, such that $V'W' \subset U$. Then $V = f_{D(a,b)}^{-1}(V')$ and $W = f_{D(a,b)}^{-1}(W')$ are the required sets.

Case II. $a = b = ab$. If this point is in D°, we are done since D is iseomorphic to K_n or K_n', so suppose it is on the boundary of D. Choose $\epsilon > 0$ such that $B(a, \epsilon) \subset U$. The components of $X - \{a\}$ form a null sequence at most, so let $C_1, \dots, C_n$ be those that meet $X - B(a, \epsilon/2)$. Some of these may contain a cyclic element D_i containing a. For each of those that do, there exists a positive $\delta_i < \epsilon$ such that

$$[B(a, \delta_i) \cap D_i][B(a, \delta_i) \cap D_i] \subset B(a, \epsilon) \cap D_i$$

since D_i is a topological semilattice. Also by Lemma 2, there exists a positive $\delta_i' \leq \delta_i$ such that for $x \in C_i - D_i$ both $P(D_i, x)$ in $B(a, \delta_i')$ implies $x \in B(a, \delta_i)$ and $x \in B(a, \delta_i')$ implies $P(D_i, x) \in B(a, \delta_i)$.

Then choose $\delta_i'' \leq \delta_i$ so that $x \in (C_i - D_i) \cap B(a, \delta_i'')$ implies $P(D_i, x) \in B(a, \delta_i)$. If $y, z \in B(a, \delta_i'') \cap C_i$, either (i) y and z are in the same component of $C_i - D_i$, in which case the choice of δ_i'' implies $yz \in B(a, \delta_i)$ since $P(D_i, yz) = P(D_i, y) = P(D_i, z) \in B(a, \delta_i')$, or (ii) $yz = P(D_i, y) \cdot P(D_i, z)$, which by the second implication is a product of two elements of $B(a, \delta_i) \cap D_i$, and hence in $B(a, \epsilon) \cap D_i$. Thus

$$[B(a, \delta_i'') \cap C_i][B(a, \delta_i'') \cap C_i] \subset B(a, \epsilon) \cap C_i.$$

Now for those C_i not containing such a D_i, each must contain a cut point x_i of itself in $B(a, \epsilon/2)$ such that $C(a, x_i) \subset B(a, \epsilon/2)$. Again by Lemma 2, there is a positive $\delta_i < \epsilon/2$ such that for x in $C_i - C(a, x_i)$, $P(C(a, x_i), x) \in B(a, \delta_i)$ implies $x \in B(a, \epsilon/2)$ and $x \in B(a, \delta_i)$ implies $P(C(a, x_i), x) \in B(a, \epsilon/2)$. Choose δ_i', $0 < \delta_i' < \delta_i$, so that $x \in B(a, \delta_i') \cap [C_i - C(a, x_i)]$ implies $P(C(a, x_i), x) \in B(a, \delta_i)$. By an argument similar to that of the preceding paragraph, one shows that

$$[B(a, \delta_i') \cap C_i][B(a, \delta_i') \cap C_i] \subset B(a, \epsilon) \cap C_i.$$

Finally by a technique like that above choose δ_0 so that $y \in B(a, \delta_0)$ and in the component of 0 in $X - \{a\}$ implies $ay \in B(a, \epsilon)$. Let δ be the smallest of all the $\delta_0, \delta_i, \delta_i'$, and δ_i''. We claim that $B(a, \delta)^2 \subset B(a, \epsilon)$. For if $y, z \in B(a, \delta)$ are in the same component of $X - \{a\}$, the above two paragraphs show that $yz \in B(a, \epsilon)$. If y and z are in different components of $X - \{a\}$ and neither of these components contain 0, then $yz = a$, while if the component of, say, y contains 0, then $yz = ya \in B(a, \epsilon)$.

Case III. $a = ab \neq b$. This will be subdivided into three subcases:

(i) Suppose $D(a, b) = a$, so that a separates b from 0, and also that $C(a, b)$ contains a true cyclic element D containing a. Then $P(D, b) \neq a$ and $ab = aP(D, b)$, so there exist disjoint relatively open subsets V' and W' of D such that $a \in V'$, $P(D, b) \in W'$, and $V'W' \subset U \cap D$. By a technique similar to that of Case II, choose a $\delta > 0$ such that if y is in $B(a, \delta)$ and the component of 0 in $X - \{a\}$ then $ya \in U$. Let $V = f_D^{-1}(V') \cap B(a, \delta)$ and $W = f_D^{-1}(W')$. Then for $y \in V$ and $z \in W$, either y is in the component of 0 in $X - \{a\}$, in which case $yz = ya \in U$, or y is not in the component of 0 in $X - \{a\}$, in which case $yz = P(D, y) \cdot P(D, z) \in V'W' \subset U$.

(ii) Suppose $D(a, b) = a$, but $C(a, b)$ contains no true cyclic element containing a. By Case II, there is an open set W' such that $W'W' \subset U$. Also $C(a, b)$ contains a cut point x with the property that $C(a, x) \subset W'$. Let V be the component of b in $X - \{x\}$, and let W be W' intersected with the component of a in $X - \{x\}$. For $y \in V$ and $z \in W$, $yz = xz \in W'W' \subset U$.

(iii) Suppose $D(a, b) \neq a$. Then $D(a, b)$ is a true cyclic element containing a which separates any two of a, b, and 0, so that $P(D(a, b), 0) \neq a$ and $P(D(a, b), b) \neq a$. Choose V' and W' disjoint relatively open subsets of $D(a, b)$ such that $P(D(a, b), b) \in W'$, $a \in V'$, $V'W' \subset U \cap D(a, b)$, but neither contains $P(D(a, b), 0)$. Then $V = f^{-1}(V')$ and $W = f^{-1}(W')$ are the required sets, for $y \in V$ and $z \in W$ imply

$$yz = P(D(a, b), y)P(D(a, b), z) \in V'W' \subset U.$$

This completes the proof of the main theorem.

Corollary. *Every retract of I^2 admits a semilattice.*

Proof. Borsuk [2] has characterized retracts of I^2 as locally connected continua which do not separate the plane. Whyburn [8] in turn has characterized these as those locally connected continua in the plane such that every true cyclic element is a simple closed curve with interior, i.e. a two-cell. From these results the Corollary follows.

BIBLIOGRAPHY

1. K. Borsuk, *Sur les retracts*, Fund. Math. **17** (1931), 164.

2. ———, *Theory of retracts*, Monografie Mat., Tom 44, PWN, Warsaw, 1967. MR **35** #7306.

3. V. Knight, *A continuous partial order for Peano continua*, Pacific J. Math. **30** (1969), 141–153. MR **39** #7570.

4. C. Kuratowski and G. T. Whyburn, *Sur les éléments cycliques et leurs applications*, Fund. Math. **16** (1930), 305–331.

5. J. D. Lawson and W. W. Williams, *Topological semilattices and their underlying spaces*, Semigroup Forum 1 (1970), no. 3, 209–223. MR **42** #3221.

6. A. D. Wallace, *Acyclicity of compact connected semigroups*, Fund. Math. **50** (1961/62), 99–105. MR **24** #A2373.

7. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloq. Publ., vol. 28, Amer. Math. Soc., Providence, R. I., 1942. MR **4**, 86.

8. ———, *Concerning the structure of the continuous curve*, Amer. J. Math. **50** (1928), 167–194.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LOUISVILLE, LOUISVILLE, KENTUCKY 40208