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ŁOŚ' THEOREM AND THE BOOLEAN PRIME IDEAL THEOREM IMPLY THE AXIOM OF CHOICE

PAUL E. HOWARD

ABSTRACT. In this paper it is shown that Łoś' theorem and the Boolean prime ideal theorem imply the axiom of choice. The possibility of eliminating the use of the Boolean prime ideal theorem from the proof is also discussed.

1. The results of this paper are proved in Zermelo-Fraenkel set theory without the axiom of choice. We denote this theory by ZF. Let AC and BPI denote the following statements in the theory ZF:

AC (The axiom of choice). *For every collection* X *of nonempty sets, there is a function* f *with domain* X *such that* $(\forall y \in X)(f(y) \in y)$. *($f$ is called a choice function for* X.)

BPI (The Boolean prime ideal theorem). *Every Boolean algebra has a prime ideal.*

Now suppose that $M = \langle A, R_j \rangle_{j \in J}$ is a relational system, i.e. A is a set and for each $j \in J$, R_j is a finitary relation on A. Let X be any set and let $\mathfrak{F}$ be a filter in the Boolean algebra of all subsets of X. We denote by A^X the set of all functions from X to A, and by $A^X/\mathfrak{F}$ we mean the set of equivalence classes of A^X under the relation $\cong$ defined by

$$f_1 \cong f_2 \quad \text{iff } \{t \in X \mid f_1(t) = f_2(t)\} \in \mathfrak{F}.$$

If $f \in A^X$, let $[f]$ denote the equivalence class of f. For each $j \in J$ we define the relation R_j' as follows: Suppose R_j is an n-ary relation on A, then R_j' is the n-ary relation on $A^X/\mathfrak{F}$ given by

$$R_j'([f_1], \cdots, [f_n]) \quad \text{iff } \{t \in X \mid R_j(f_1(t), \cdots, f_n(t))\} \in \mathfrak{F}.$$

One can easily show that R is well defined. Finally we define the reduced ultra power

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$$M^X/\mathfrak{F} = \langle A^X/\mathfrak{F}, R_j' \rangle_{j \in J}.$$

Using the axiom of choice, Łoś' [1, p. 104] has shown

LT (Łoś' theorem). *If* $M = \langle A, R_j \rangle_{j \in J}$ *is a relational system,* X *any set and* $\mathfrak{F}$ *an ultrafilter in the power set of* X*, then* $M^X/\mathfrak{F}$ *and* M *are elementarily equivalent.*

We assume that the first order language associated with M has as nonlogical symbols an n-ary relation symbol $\mathbf{R}_j$ for each $j \in J$ (where R_j is n-ary) and as nonlogical symbols $\exists, \vee, \neg$. The other logical symbols are then abbreviations: $\forall$ for $\neg\exists\neg$ etc.

To prove LT, one proves by induction on formulas $\Phi(x_1, \cdots, x_n)$ in the first order language associated with M that for any $[f_1], \cdots, [f_n] \in A^X/\mathfrak{F}$

(1) $\quad M^X/\mathfrak{F} \models \Phi([f_1], \cdots, [f_n])$ iff $\{t \in X | M \models \Phi(f_1(t), \cdots, f_n(t))\} \in \mathfrak{F}$.

The axiom of choice is necessary at only one place in the proof; namely in proving the implication from right to left in (1) in the case of the existential quantifier.

2. The purpose of this paper is to prove the following

Theorem. LT + BPI $\Rightarrow$ AC.

Proof. Suppose X is a collection of nonempty sets. We are going to show that X has a choice function so we may assume without loss of generality that the elements of X are pairwise disjoint and that for each $y \in X$, $y \cap X = \varnothing$.

Let $A = X \cup (\bigcup X)$ and define the binary relation R on A by

$$\begin{array}{ll} tRy & \text{if } y \in X \text{ and } t \in y, \\ tRt & \text{if } t \in \bigcup X, \\ \text{not}\,(zRw) & \text{otherwise.} \end{array}$$

If X has a choice function we are done, so suppose that X has no choice function. Let $M = \langle A, R \rangle$. We are going to apply Łoś' theorem to the structure $M^X/\mathfrak{F}$ where $\mathfrak{F}$ is an ultrafilter obtained in the following way: Let $I = \{z \subseteq X | z$ has a choice function$\}$. It is clear that I is an ideal and since X has no choice function, I is a proper ideal. Using BPI we extend I to a maximal (proper) ideal I'. Then let $\mathfrak{F} = \{X \backslash z | z \in I'\}$ where $X \backslash z$ denotes $\{y \in X | y \notin z\}$. One can easily verify that $\mathfrak{F}$ is an ultrafilter.

By Łoś' theorem $M^X/\mathfrak{F}$ and M are elementarily equivalent, and by the definition of R we have $M \models \forall y \exists t(tRy)$ and therefore $M^X/\mathfrak{F} \models \forall y \exists t(tR'y)$. In particular if we let 1_X be the identity function on X and take $y = [1_X]$, we get that for some function $f\colon X \to A$, $[f]\, R'\, [1_X]$ so $\{y \mid f(y)\, R\, 1_X(y)\} \in \mathfrak{F}$, that is $\{y \mid f(y) \in y\} \in \mathfrak{F}$, therefore $\{y \mid f(y) \notin y\} \in I'$. But the set $\{y \mid f(y) \in y\}$ has a choice function and is therefore in I' which contradicts the choice of I' as a proper ideal. Hence the assumption that X has no choice function leads to a contradiction and the proof of the Theorem is complete.

One final comment on eliminating BPI from the Theorem: Consider the statement

EF. *There exists a nonprincipal ultrafilter.*

One can easily show $\neg$ EF implies LT since if $\mathfrak{F}$ is a principal ultrafilter on $P(X)$, then $M^X/\mathfrak{F}$ is isomorphic to M. To the knowledge of the author, it is unknown if $\neg$ EF is consistent with ZF. If one could show this consistency, then one would have the consistency of ZF + LT + $\neg$ AC. Thus the elimination of BPI from the Theorem would be impossible.

REFERENCE

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DEPARTMENT OF MATHEMATICS, EASTERN MICHIGAN UNIVERSITY, YPSILANTI, MICHIGAN 48197