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CHARACTERISTIC SUBMANIFOLDS OF FIXED POINT FREE INVOLUTIONS[1]

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ABSTRACT. According to López de Medrano, a manifold with a free involution is determined by its Browder-Livesay index and the normal cobordism class of its orbit space. Here we study the Browder-Livesay indices of some cobordant involutions, namely, the characteristic submanifolds of a fixed involution.

In [2] and [3], studying fixed point free involutions on homotopy spheres, W. Browder and G. R. Livesay defined an index σ measuring the obstruction to finding an invariant embedded sphere of codimension 1. For a closed orientable manifold M^{4k-1} with a fixed point free involution T which preserves orientation, the Browder-Livesay index $\sigma(T, M)$ is defined as the signature of a bilinear form on the homology of a characteristic submanifold; a characteristic submanifold W is a closed, orientable submanifold of codimension 1 whose complement has two components A and B such that $B = TA$ and $\mathrm{Cl}\, A \cap \mathrm{Cl}\, B = W$. Since characteristic submanifolds arise as $f^{-1}RP^{n-1}$ where $f: M/T \to RP^n$ (real projective space) is the classifying map of the Z_2 bundle $M \to M/T$, characteristic submanifolds exist and any two are equivariantly cobordant in $M \times I$. It follows that σ does not depend on choice of W [1].

However, if M_0 and M_1 are characteristic submanifolds of (T, X^{4k}), in general $\sigma M_0 \neq \sigma M_1$; if X is S^{4k}, $\sigma M_0 - \sigma M_1 = 2\sigma(T \times 1;\ S \times I, M_0 - M_1)$ [1] and if M_0 and M_1 are spheres, (T, M_0) and (T, M_1) are equivariantly diffeomorphic iff $\sigma M_0 = \sigma M_1$ [2], [3].

Here we derive a formula for $\rho(M_0, M_1) = \sigma M_0 - \sigma M_1 - 2\sigma(X \times I, M_0 - M_1)$ and show that it lies between $\pm \dim H_{2k}X$ and does assume all these values. We also give an algebraic condition for a subspace of $H_{2k}X$ to be $H_{2k}A$ where $A \cap TA$ is a characteristic submanifold of X.

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Throughout this paper, all manifolds are smooth or PL, compact and orientable, all involutions are smooth or PL and have no fixed points. Homology groups have rational coefficients, except where otherwise stated. B will be used for any bilinear form on a subspace of $H_{2k}N^{4k}$ given by $B(x, y) = x \cdot Ty$, the intersection number of x and Ty in N; the annihilator, under B, of a subspace C will be denoted $\mathfrak{A}C$.

Suppose S is an orientation reversing free involution on a compact manifold Y^{4k+1}:

Definition. A characteristic submanifold N^{4k} of (S, Y) is an invariant submanifold of Y, meeting ∂Y transversely (in ∂N) for which there are compact submanifolds U and V of Y with $U \cup V = Y$, $U \cap V = N$ and $V = SU$. Then ∂N will be a characteristic submanifold of $(S, \partial Y)$ splitting ∂Y into $(U \cap \partial Y) \cup (V \cap \partial Y)$.

Definition. The Browder-Livesay index $\sigma(S; Y, N)$ is the signature of the symmetric bilinear form $B(x, y) = x \cdot Sy$ defined on $\ker(H_{2k}N \to H_{2k}U)$. In fact, given ∂N this is independent of N.

Now suppose (T, M^{4k-1}) is a given characteristic submanifold of $(T, X^{4k}) = (S|\partial Y, \partial Y)$. Then M can be extended to a characteristic submanifold N^{4k} of Y, splitting Y into $U \cup V$; let $A = U \cap X$ and $B = V \cap X$. Notice that T/X reverses and T/M preserves the orientation.

Suppose all maps in the diagram are induced by the inclusions. Let $K = u^{-1}\operatorname{im} i \cap v^{-1}\operatorname{im} j \subseteq H_{2k}N$, and $L = \operatorname{im} a \cap \operatorname{im} b \subseteq H_{2k}Y$.

$$\begin{tikzcd}
 & H_{2k}N \arrow[r, "v"] \arrow[d, "u"] \arrow[dl] & H_{2k}V \arrow[d, "l"] & H_{2k}B \arrow[l, "j"'] \arrow[dl, "b"] \\
H_{2k}\partial U \arrow[r] & H_{2k}U \arrow[r, "k"] & H_{2k}Y & \\
 & H_{2k}A \arrow[u, "i"] \arrow[ul] \arrow[ur, "a"'] & &
\end{tikzcd}$$

Define a bilinear form ρ on L as follows: if $x_i = a(s_i) = b(t_i)$, $i = 1, 2$, set $\rho(x_1, x_2) = s_1 \cdot Tt_2$, where the intersection is taken in X.

Lemma. *ρ is well defined (i.e. is independent of the choices of s_i and t_i) and is induced from the bilinear form $-B$ on K.*

Proof. First we show there is an induced form. By the Mayer-Vietoris

sequence for Y, the sequence

$$0 \longrightarrow \ker ku \cap K \longrightarrow K \longrightarrow L \longrightarrow 0$$

is exact. The kernel of $-B$ restricted to K is $\mathfrak{A}K \cap K$, where

$$\mathfrak{A}K = \mathfrak{A}(u^{-1} \operatorname{im} i \cap v^{-1} \operatorname{im} j) = \mathfrak{A}(u^{-1} \operatorname{im} i) + \mathfrak{A}(v^{-1} \operatorname{im} j) \supseteq \ker v + \ker u,$$

since $\mathfrak{A} \ker v = u^{-1} \operatorname{im} i$. But by the Mayer-Vietoris sequence for Y, $\ker u + \ker v = \ker ku$; therefore $\ker ku \cap K \subseteq \ker -B|K$, so that $-B$ induces a bilinear form on L.

To show that this induced form is ρ, let x_1 and $x_2 \in L$, and pick s_i and t_i, $i = 1, 2$, such that $x_i = as_i = bt_i$; then there exist $m_i \in K$ such that $um_i = is_i$ and $vm_i = jt_i$. If $-s_1 \cdot Tt_2 = m_1 \cdot Sm_2$, since $x_i = kum_i$, then ρ is the induced form. In fact, since $\partial U = A \cup N$, intersection numbers are equal whether taken in A and N or in ∂U. Regarded as elements of $H_{2k}\partial U$, $(s_1 - m_1)$ and $T(t_2 - m_2) \in \ker(H_{2k}\partial U \longrightarrow H_{2k}U)$, hence their intersection number is 0. Since $H_{2k}A$ and $H_{2k}N$ are orthogonal in $H_{2k}\partial U$, $s_1 \cdot Tt_2 + m_1 \cdot Sm_2 = 0$ i.e. $s_1 \cdot Tt_2 = -m_1 \cdot Sm_2$. Therefore ρ is the induced form and hence is well defined.

Theorem 1. $\sigma(T, M) = 2\sigma(S; Y, N) -$ signature ρ.

Proof. Since $(T, M) = \partial(S, N)$ and S has no fixed points, $\sigma(T, M) =$ signature B on $H_{2k}N$, where $B(x, y) = x \cdot Sy$ [4], [5].

Notice that if F is a bilinear form on a vector space W, and if C is a subspace for which $C \subseteq \mathfrak{A}C$, then signature $F =$ signature $F|\mathfrak{A}C$. For by replacing C by $C + \ker F$, we may assume $\ker F \subseteq C$; then W can be written as $C \oplus D \oplus E$ where $\mathfrak{A}C = C \oplus D$ and $\mathfrak{A}D = C \oplus E$. Now $\operatorname{sig} F = \operatorname{sig} F|C \oplus E + \operatorname{sig} F|D$, but since the annihilator of C in $C \oplus E$ is C, $\operatorname{sig} F|C \oplus E = 0$, and since $C \subseteq \ker F|D \oplus C$, $\operatorname{sig} F = \operatorname{sig} F|\mathfrak{A}C$.

By Poincaré duality, the kernel of B on $H_{2k}N$ is $\ker p$ where $p: H_{2k}N \longrightarrow H_{2k}N, M$ is induced by the projection. Let $C = (\ker u + \ker p) \cap \mathfrak{A} \ker u$, then since $C \subseteq \mathfrak{A}C$, $\operatorname{sig} B = \operatorname{sig} B|\mathfrak{A}C$; but $\mathfrak{A}C = \mathfrak{A} \ker u + \ker u$ where these two terms are orthogonal, so that

$$\operatorname{sig} B = \operatorname{sig} B|\ker u + \operatorname{sig} B|\mathfrak{A} \ker u.$$

Let $W = \mathfrak{A} \ker u$ and let $J = \ker B|W$. Consider $D = \mathfrak{A} \ker v \cap W$ and $E = \ker v + J$, where $E \subseteq W$ since $\ker v \subseteq \mathfrak{A} \ker u$. Then $\mathfrak{A}E$ (in W) is $\mathfrak{A} \ker v \cap \mathfrak{A}J \cap W = D$ and since $E \supseteq J$, $E = \mathfrak{A}\mathfrak{A}E = \mathfrak{A}D$, so that $\mathfrak{A}(E \cap D) =$

$E + D \supseteq E \cap D$. Therefore $\operatorname{sig} B|W = \operatorname{sig} B|\mathfrak{A}(E \cap D) = \operatorname{sig} B|E \oplus D$, which is equal, since E and D are orthogonal, to $\operatorname{sig} B|E + \operatorname{sig} B|D$.

Since $J \subseteq \ker B|E$, $\operatorname{sig} B|E = \operatorname{sig} B|\ker v$, and S, preserving orientation on N, also preserves intersection numbers, so that $\operatorname{sig} B|E = \operatorname{sig} B|\ker u$.

Finally, $D = \mathfrak{A} \operatorname{Ker} v \cap \mathfrak{A} \ker u = u^{-1} \operatorname{im} i \cap v^{-1} \operatorname{im} j = K$; since ρ is induced from $-B|K$, by dividing out by part of the kernel, $B|D = \operatorname{sig} -\rho$. Therefore

$$\sigma(T, M) = 2 \operatorname{sig} B|\ker u - \operatorname{sig} \rho = 2\sigma(S;\ Y, N) - \operatorname{sig} \rho.$$

Note. Since $\sigma(T, M)$ and signature ρ are determined by the embedding of ∂N in ∂Y, $\sigma(S;\ Y, N)$ does not depend on N, but only on ∂N, and will be denoted $\sigma(S;\ Y, \partial N)$.

Corollary. *Suppose M_0 and M_1 are two characteristic submanifolds of (T, X^{4k}) where T reverses orientation and X is closed. Then*

$$\sigma M_0 - \sigma M_1 = 2\sigma(T \times 1;\ X \times I,\ M_0 - M_1) - \operatorname{sig} \rho.$$

In particular, when $H_{2k}X = 0$, $\sigma M_0 - \sigma M_1 = 2\sigma(T \times 1;\ X \times I,\ M_0 - M_1)$ [1] and when X is a homotopy sphere, this is a multiple of 16 since the bilinear forms are even and unimodular. However, in general $\sigma M_0 - \sigma M_1$ need not be even; here is an example where $\sigma M_0 = 0$ and $\sigma M_1 = 1$.

Let $X = S^{2k} \times S^{2k}$ and $T = A \times 1$ where A is the antipodal map.

Let $M_0 = S^{2k-1} \times S^{2k}$; a characteristic submanifold of M_0 is $W_0 = S^{2k-2} \times S^{2k}$ and since $H_{2k-1}W_0 = 0$, $\sigma M_0 = 0$.

Let $M_1 = \{(x, y) \in S^{2k} \times S^{2k} | x \cdot y = 0\}$. Then $p: M_1 \to S^{2k}$ (projection on the first factor) is a fibre bundle with fibre S^{2k-1}. S^{2k} has a characteristic submanifold S dividing it into two discs, D_+ and D_-. Then $W_1 = p^{-1}S$ is a characteristic submanifold of M_1, dividing it into $p^{-1}D_+ \cup p^{-1}D_-$. Since the bundle is trivial when restricted to S, $H_{2k-1}W_1 = Q \oplus Q$, where one generator F is represented by the fibre, and the other, C, is represented by the cross-section $S^{2k-1} \to M_1$, $x \to (x, \alpha(x))$ where $\alpha(0, x_1, \ldots, x_{2k}) = (0, -x_2, x_1, \ldots, -x_{2k}, x_{2k-1})$. Now $TC = C$ since $\alpha \sim A\alpha$, and $TF = -F$. $p^{-1}D_+ = D_+ \times S^{2k-1}$ and the map $H_{2k-1}W_1 \to H_{2k-1}p^{-1}D_+$ is determined by $C \to F$ and $F \to F$. Therefore $\ker H_{2k-1}W_1 \to H_{2k-1}p^{-1}D_+ = Q$ and a generator is $C - F$. Since

$$(C - F) \cdot T(C - F) = (C - F) \cdot (C + F) = 2 > 0,$$

$\sigma M_1 = 1$.

In this example $L = Q \oplus Q = H_{2k}\partial(X \times I)$ and $\operatorname{sig} \rho = 1$; $\sigma(T \times 1) = 0$.

Proposition. *If T is an orientation reversing involution on a manifold X^{4k} with boundary, then given $(M_0, \partial M_0)$ a characteristic submanifold of $(X, \partial X)$ where ∂M_0 divides ∂X into $P \cup TP$, then there is a characteristic submanifold $(M_1, \partial M_1)$ with $\sigma M_1 = \sigma M_0$, such that $\partial M_1 = \partial M_0$, $H_r(X, A_1; Z) = 0$ for $r < 2k$ and $H_r(A_1, P; Q) \to H_r(X, P; Q)$ is a monomorphism for $r \leq 2k$.*

The proof is by equivariant handle exchange between A and B to make the pair (A, M) $2k - 1$ connected.

Suppose inductively that $H_p(A_0, M_0; Z) = 0$ for all $p <$ some fixed $r \leq 2k$. Then the Hurewicz map $\pi_r(A_0, M_0) \to H_r(A_0, M_0; Z)$ is surjective. Let $\{x_1, \ldots, x_n\} \subset \pi_r(A_0, M_0)$ represent a basis for $H_r(A_0, M_0; Z)$. If $r < 2k$, x_1 can be represented by an embedding $D^r, S^{r-1} \subset A_0, M_0$ which is disjoint from Tx_1 [8]. Let H be a tubular neighborhood of D^r, S^{r-1} in A_0, M_0, so that $H = D^r \times D^{4k-r}$ and $H \cap M_0 = S^{r-1} \times D^{4k-r}$, which is disjoint from TH; and let A_1, B_1 and M_1 be the manifolds resulting from exchanging H and TH, that is $A_1 = A_0 - H \cup TH$, $B_1 = B_0 \cup H - TH$ and

$$M_1 = M_0 - (S^{r-1} \times D^{4k-r}) \cup (D^r \times S^{4k-r-1})$$
$$- T(S^{r-1} \times D^{4k-r}) \cup T(D^r \times S^{4k-r-1}).$$

Now (A_0, M_0) is equivariantly cobordant to (A_1, M_1) in $X \times I$; in particular let K be a tubular neighborhood of (D^r, S^{r-1}) in $(A_0 \times I, M_0 \times 1)$, and set

$$U = A_0 \times I - K \cup TK,$$

$$V = (T \times 1)U = B_0 \times I \cup K - TK \quad \text{and} \quad N = U \cap V.$$

Then clearly N is a characteristic submanifold of $X \times I$ and (U, N) is a cobordism between (A_0, M_0) and (A_1, M_1).

Then $H_r(A_1, M_1; Z)$ is generated by $\{x_2, \ldots, x_n\}$ and $H_*(A_1, M_1; Z) = H_*(A_0, M_0; Z)$ in dimensions $< r$. Since N is the trace of surgery on x_1 and Tx_1, $H_{2k}(N, M_0; Z) = 0$ and $H_{2k}M_0 \to H_{2k}N$ is an epimorphism. Since $M_0 \subset \partial N$ and intersection numbers are 0 on boundaries, $\sigma M_0 - \sigma M_1 = \operatorname{sig} B|H_{2k}N$ [4], [5] $= 0$.

When $r = 2k$, x_1 can still be represented by an embedding of $D^{2k}, S^{2k-1} \to A_0, M_0$ [7, p. 39] disjointly from Tx_1. If x_1 has infinite order in $H_{2k}(A_0, M_0; Z)$, surgery on x_1 will reduce the rank of $H_{2k}(A_0, M_0; Q)$ if and only if $\partial Tx \neq 0$ in $H_{2k-1}(A_0, P; Q)$. In this case $H_{2k}M_0$ is isomorphic to $H_{2k}N$, because in the exact sequence

$$0 \to H_{2k}M_0 \to H_{2k}N \to Qx \oplus QTx \xrightarrow{\partial} H_{2k-1}M_0,$$

∂ is a monomorphism since the composition

$$Qx \oplus QTx \to H_{2k-1}M_0 \to H_{2k-1}A_0, P \oplus H_{2k-1}B_0, TP$$

sends (ax, bTx) into $(b\partial Tx, a\partial x)$. Therefore, as before, $\sigma M_0 = \sigma M_1$.

We can assume, then, that M is a characteristic submanifold with $H_*(A, M; Z) = 0$ in dimensions $< 2k$, and for which the composition

$$H_{2k}A, M \to H_{2k-1}M \xrightarrow{T} H_{2k-1}M \to H_{2k-1}A, P$$

is 0, that is, since T is an isomorphism, that the maps

$$H_{2k}A, M \to H_{2k-1}B, TP \quad \text{and} \quad H_{2k}B, M \to H_{2k-1}A, P$$

are 0. It follows that in the Mayer-Vietoris sequence

$$H_{2k}X, P \to H_{2k-1}M \to H_{2k-1}A, P \oplus H_{2k-1}B \to H_{2k-1}X, P,$$

$$\operatorname{im}(H_{2k}X, P \to H_{2k-1}M) = \ker(H_{2k-1}M \to H_{2k-1}B).$$

Taking duals we see that $\operatorname{im} i = \operatorname{im} j$

$$\begin{tikzcd}
H^{2k}X, TP \arrow[r] \arrow[dr, "i"'] & H^{2k}B, TP \arrow[r] \arrow[d, "j"] & H^{2k+1}X, B \arrow[d, "\cong"'] \\
 & H^{2k}M \arrow[r] & H^{2k+1}A, M
\end{tikzcd}$$

so that h is an epimorphism, from which it follows that $H_{2k}(A, P) \to H_{2k}(X, P)$ is a monomorphism. From the 5 lemma $H_*(A, P) \to H_*(X, P)$ is a monomorphism in all dimensions $\leq 2k$.

Theorem 2. *Let T be an orientation reversing involution of X^{4k} where X is a simply connected closed manifold and $k > 1$. Let C be a (vector) subspace of $H_{2k}(X; Q)$. Then there is a characteristic submanifold M dividing X into $A \cup B$ with $H_{2k}A = C$, if and only if $C = \mathfrak{A}C$, that is, since B is nonsingular, if and only if $C \cdot TC = 0$ and* $\dim C = \frac{1}{2} \dim H_{2k}X$.

Proof. $\Leftarrow$ Clearly if $C = H_{2k}A$, $C \cdot TC = 0$. Since the Alexander dual of $\operatorname{im}(H_{2k}A \to H_{2k}X)$ is $\operatorname{im}(H^{2k}X, B \to H^{2k}X)$ which is isomorphic to $\operatorname{im}(H_{2k}X \to H_{2k}X, A)$, and since by exactness,

$$\dim \operatorname{im}(H_{2k}A \to H_{2k}X) + \dim \operatorname{im}(H_{2k}X \to H_{2k}X, A) = \dim H_{2k}X,$$

$2 \dim C = \dim H_{2k}X$.

$\Rightarrow$ Suppose f is an immersion: $M^{2k} \to X$ with $f_*[M] = x \in H_{2k}X$. Then there is an immersion g of $M + M$, approximating $f + f$, with g disjoint from

Tg, because if p is the projection: $X \to X/T$, then the self-intersection number of p_*2x will have no component in the generator of $\pi_1 X/T$, so that p_*2x can be represented without any intersections arising from intersections of x and Tx [7, p. 45]. We can then represent a basis for C by immersed manifolds M_i each disjoint from all TM_j [6]. Since X is 1-connected, the image of each M_i can be assumed 1-connected. Let U be a regular neighborhood of the union of the M_i disjoint from TU. $X - U - TU$ has a closed characteristic submanifold M dividing it into $Y \cup TY$, where $\partial Y = \partial U$, which can be chosen so that $H_r Y, M = 0$, $r < 2k$, and $H_r(Y, \partial U) \to H_r(Y \cup TY, \partial U)$ is a monomorphism for $r \leq 2k$. Set $A = U \cup Y$.

Now image $H_{2k}A \to H_{2k}X = C$ because image $H_{2k}A \to H_{2k}X$ is contained in its annihilator under B and contains C which equals $\mathfrak{A}C$. Therefore it remains to show that $H_{2k}A \to H_{2k}X$ is a monomorphism.

Now a is a monomorphism, hence so are b and c. Thus $\ker d \subset e(\ker f)$; but since g is a monomorphism, $\ker f \subset \ker e$; therefore $\ker d = 0$. h is a monomorphism because

$$\ker h = \operatorname{im}(H_{2k+1}X, X - TU \to H_{2k}X - TU)$$

and $H_{2k+1}X, X - TU = H^{2k-1}TU$ which is 0 since each $H_1 M = 0$. Therefore hd is a monomorphism: $H_{2k}A \to H_{2k}X$.

$$\begin{tikzcd}
H_{2k}Y \cup TY, \partial U \arrow[d] & & H_{2k}Y, \partial U \arrow[ll, "a"'] \arrow[d] \\
H_{2k}Y \cup TY, \partial U \oplus H_{2k}U, \partial U \arrow[d, "\cong"] & & H_{2k}Y, \partial U \oplus H_{2k}U, \partial U \arrow[ll, "b"'] \arrow[d, "\cong"] \\
H_{2k}X \cup TU, \partial U & & H_{2k}Y \cup U, \partial U \arrow[ll, "c"'] \\
H_{2k}X - TU \arrow[u] \arrow[d, "h"'] & & H_{2k}Y \cup U \arrow[ll, "d"'] \arrow[u] \\
 & H_{2k}\partial U \arrow[ul, "f"] \arrow[ur, "e"'] & \\
H_{2k}X & & H_{2k}U \arrow[ll, "g"'] \arrow[uu]
\end{tikzcd}$$

As an application we have

Theorem 3. *Let T be an orientation reversing involution of X^{4k} where X is a simply connected closed manifold and $k > 1$. Then there is a characteristic submanifold M_0 of X such that given $|r| \leq n = \dim H_{2k}X$, there is another characteristic submanifold of M_1 of X for which*

$$\operatorname{sig} \rho(M_0, M_1) = \sigma(M_0) - \sigma(M_1) - 2\sigma(X \times T_1 M_0 - M_1) = r.$$

Proof. First we find a subspace C of $H_{2k}X$ for which $C \oplus TC = H_{2k}X$ and $C \cdot TC = 0$, then C can be realized as $H_{2k}(A)$ (Theorem 2); set $M_0 = \partial A$. Let $V = \{x : x = Tx\}$ and $W = \{x : x = -Tx\} \subset H_{2k}X$, then $H_{2k}X = V \oplus W$, but since T has Lefschetz number 0, and reverses orientation, trace $T : H_{2k}X \to H_{2k}X = 0$, so that $\dim Y = \dim W = \frac{1}{2} \dim H_{2k}X$, and $V = \mathfrak{A}V$ and $W = \mathfrak{A}W$. Therefore $H_{2k}X$ has a symplectic basis $\{v_i, w_i : i = 1, \ldots, n/2\}$, that is, $v_i \cdot v_j = 0 = w_i \cdot w_j$ for all i and j and $v_i \cdot w_j \neq 0$ only if $i = j$ and $v_i \cdot w_i = 1$, with $Tv_i = v_i$ and $Tw_i = -w_i$. Let C be the space generated by $\{v_i + w_i : i = 1, \ldots, n/2\}$; then $C \cap TC = 0$, $C + TC = H_{2k}X$ and $C \cdot TC = 0$.

Given $n/2$ pairs of rational numbers (α_i, β_i) with each $\alpha_i < \beta_i$, let C_1 be the subspace of $H_{2k}X$ generated by $\{\alpha_i v_i + \beta_i w_i\}$; like C, C_1 can be realized by A; let $M_1 = \partial A_1$. We wish to choose $\{\alpha_i, \beta_i\}$ so that $\operatorname{sig} \rho(M_0, M_1) = r$.

Now ρ is defined on $(C + C_1) \cap T(C + C_1)$ which is $H_{2k}X$. Let X_i denote the subspace of $H_{2k}X$ generated by v_i and w_i; then the family of X_i's are mutually orthogonal under B, hence also under ρ, since

$$\rho(c + c_1, c' + c_1') = B(c, c') - B(c_1, c_1').$$

Consequently $\operatorname{sig} \rho = \Sigma \operatorname{sig} \rho | X_i$. But

$$\rho | X_i = \begin{vmatrix} 2\beta_i/(\beta_i - \alpha_i) & 0 \\ 0 & 2\alpha_i/(\beta_i - \alpha_i) \end{vmatrix};$$

since $\alpha_i < \beta_i$, $\operatorname{sig} \rho | X_i = \operatorname{sign} \beta_i + \operatorname{sign} \alpha_i$.

Clearly then α_i and β_i can be chosen so that $\operatorname{sig} \rho | X_i$ is 2, 1, 0, −1 or −2. It follows that $\operatorname{sig} \rho$ can be any number between $-n$ and n.

Since the maximum dimension, for all M_0 and M_1 of the space L on which ρ is defined, is n, $\rho(M_0, M_1)$ can assume all possible values.

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