ABSTRACT. Suppose $f$, $h$ and $G$ are functions with values in a normed complete ring. With suitable restrictions on these functions, it is established that

$$f(x) = h(x) + \int_a^x f(u)G(u, v)$$

for $a \leq x \leq b$ only if $\int_a^x h(u)G(u, v) \Pi^x (1 + G)$ exists and is $f(x) - h(x)$ for $a \leq x \leq b$, and that

$$f(x) = h(x) + \int_a^x G(u, v)f(u)$$

for $a \leq x \leq b$ only if $\int_a^x \Pi^x (1 + G)h(u)G(u, v)h(u)$ exists and is $f(x) - h(x)$ for $a \leq x \leq b$, where $G(s, r) = G(r, s)$.

Lower case letters are used to represent functions from $R$ to $N$, and capital letters are used to represent functions from $R \times R$ to $N$, where $R$ denotes the set of real numbers and $N$ denotes a ring which has a multiplicative identity element represented by $1$ and a norm $|\cdot|$ with respect to which $N$ is complete and $|1| = 1$. For a subdivision $\{x_i\}_{i=0}^n$ of an interval $[a, b]$, we use $G_i$ and $f_i$ to denote $G(x_{i-1}, x_i)$ and $f(x_i)$, respectively.

The statement that $G \in OB^\iota$ on $[a, b]$ means there exist a subdivision $D$ of $[a, b]$ and a number $B$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of $D$, then $\sum_{i=1}^n |G_i| < B$.

The statement that $f_a^b G$ exists means there exists an element $L$ of $N$ such that, if $\epsilon > 0$, then there exists a subdivision $D$ of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of $D$, then $|L - \sum_{i=1}^n G_i| < \epsilon$. Further, $G \in OA^\iota$ on $[a, b]$ only if $f_a^b G$ exists and $f_a^b |G - \int G| = 0$.

The statement that $\int_a^b (1 + G)$ exists means there exists an element $L$
of \(N\) such that, if \(\epsilon > 0\), then there exists a subdivision \(D\) of \([a, b]\) such that, if 
\[
\{ x_i \}^{n}_{i=0} \quad \text{is a refinement of } D, \quad \text{then } |L - \Pi^n_{i=1} (1 + G_i) | < \epsilon.
\]
Further, \(G \in OM^o\) on \([a, b]\) only if \(x \Pi^y (1 + G)\) exists for \(a \leq x \leq y \leq b\) and 
\[
\int_a^b |1 + G - \Pi (1 + G)| = 0.
\]
The existence of integrals is defined similarly on intervals \([p, q]\), where \(q < p\).

The function \(h\) is quasi-continuous on \([a, b]\) only if \(\lim_{x \to p^-} h(x)\) exists for \(a < p < b\) and \(\lim_{x \to p^+} h(x)\) exists for \(a < p < b\). Further, \(G \in O L^o\) on \([a, b]\) only if \(\lim_{x \to p^-} G(x, p)\) and \(\lim_{x, y \to p} G(x, y)\) exist for \(a < p \leq b\), and \(\lim_{x \to p^+} G(p, x)\) and \(\lim_{x, y \to p} G(x, y)\) exist for \(a \leq p < b\). See B. W. Helton [2] and J. S. Mac Nerney [6] for additional background.

The theorems in this paper were suggested by a result of J. S. Mac Nerney [5, Theorem 3.4, p. 361]. However, their proofs are constructed by using techniques similar to those employed by B. W. Helton [2, §5, pp. 307–314]. There, the integral equations
\[
\int f(x) = h(x) + \int_a^x f(u) G(u, v) \quad \text{and} \quad \int f(x) = h(x) + \int_a^x G(u, v) f(u)
\]
were solved with the restriction that \(h\) have bounded variation. We now solve these integral equations with the restriction that \(h\) be quasi-continuous. The reader is also referred to related results by D. B. Hinton [4, Theorems 4.1, 4.2, 4.3, 4.4, pp. 325–327] and C. W. Bitzer [1, Theorems 4.1, 4.2, 5.9, pp. 447–451].

We now state three lemmas that are used in the development of the results in this paper.

Lemma 1. If \(G\) is a function from \(R \times R\) to \(N\) and \(G \in OB^o\) on \([a, b]\), then the following statements are equivalent:

1. \(G \in OA^o\) on \([a, b]\), and
2. \(G \in OM^o\) on \([a, b]\) [2, Theorem 3.4, p. 301].

Lemma 2. If \(H\) and \(G\) are functions from \(R \times R\) to \(N\), \(H \in OL^o\) on \([a, b]\) and \(G\) is in \(OA^o\) and \(OB^o\) on \([a, b]\), then \(GH\) and \(HG\) are in \(OA^o\) and \(OM^o\) on \([a, b]\) [3, Theorem 2, p. 494].

Lemma 3. If \(F\) and \(G\) are functions from \(R \times R\) to \(N\) belonging to \(OB^o\) on \([a, b]\), \(F \in OA^o\) on \([a, b]\) and each of \(x \Pi^y (1 + G)\) and \(\int_a^y F(u, v) \Pi^y (1 + G)\) exists for \(a \leq x < y \leq b\), then
\[
\int_a^b \left| F(x, y) - \int_x^y F(u, v) \Pi^y (1 + G) \right|
\]
is zero [2, Lemma, p. 307].
**Theorem 1.** If \( f \) and \( h \) are functions from \( R \) to \( N \), \( G \) is a function from \( R \times R \) to \( N \), \( h \) is quasi-continuous on \([a, b]\) and \( G \in OB^O \) on \([a, b]\), then the following statements are equivalent:

1. \( f \) is bounded on \([a, b]\), \( G \in OA^O \) on \([a, b]\), \( f(u)G(u, v) \in OA^O \) on \([a, b]\) and \( f(x) = h(x) + \int_a^x f(u)G(u, v) \) for \( a \leq x \leq b \), and
2. \( G \in OM^O \) on \([a, b]\) and \( \int_a^b h(u)G(u, v) \Pi^x(1 + G) \) exists and is \( f(x) - h(x) \) for \( a \leq x \leq b \).

**Proof.** (1) \( \rightarrow \) (2). Since \( G \) is in \( OA^O \) and \( OB^O \) on \([a, b]\), it follows from Lemma 1 that \( G \in OM^O \) on \([a, b]\). Suppose \( a \leq x \leq b \). If \( a = x \), then (2) follows immediately. Therefore, suppose \( a < x \). The existence of \( I(a, x) \) follows from Lemma 2, where

\[
I(r, s) = \int_r^s h(u)G(u, v) \Pi^s(1 + G)
\]

for \( a \leq r \leq s \leq b \). We now show that \( I(a, x) \) is equal to \( f(x) - h(x) \). Let \( \epsilon > 0 \).

There exists a subdivision \( D_1 \) of \([a, x]\) such that, if \( \{x_i\}_{i=0}^n \) is a refinement of \( D_1 \), then

\[
\left| I(a, x) - \sum_{i=1}^n h_{i-1}G_i \Pi^x_i(1 + G) \right| < \epsilon/3.
\]

There exists a subdivision \( D_2 \) of \([a, x]\) and a number \( B \) such that, if \( \{x_i\}_{i=0}^n \) is a refinement of \( D_2 \), then

1. \( \sum_{i=1}^n |G_i| < B \), and
2. \( \sum_{i=1}^n |h_{i-1}G_i| < B \).

Since \( G \) is in \( OM^O \) and \( OB^O \) on \([a, x]\), there exists a subdivision \( D_3 \) of \([a, x]\) such that, if \( \{x_i\}_{i=0}^n \) is a refinement of \( D_3 \) and \( 1 \leq i \leq n \), then

\[
\left| \sum_{i=1}^n \Pi^x_i(1 + G) - \prod_{k=i+1}^n (1 + G_k) \right| < \epsilon(3B)^{-1}.
\]

Since \( f(u)G(u, v) \in OA^O \) on \([a, x]\), there exists a subdivision \( D_4 \) of \([a, x]\) such that, if \( \{x_i\}_{i=0}^n \) is a refinement of \( D_4 \), then

\[
\sum_{i=1}^n \left| f_{i-1}G_i - \int_{x_{i-1}}^{x_i} f(u)G(u, v) \right| < \epsilon(3B)^{-1}.
\]

Let \( D \) denote the subdivision \( \bigcup_{i=1}^n D_i \) of \([a, x]\). Suppose \( \{x_i\}_{i=0}^n \) is a refinement of \( D \). For \( 1 \leq i \leq n \), we have that
where
\[ c_i = \int_{x_{i-1}}^{x_i} f(u)G(u, v) - f_{i-1}G_i. \]
Thus,
\[ f_i = h_i - h_{i-1} + f_{i-1}(1 + G_i) + c_i. \]
Now, by using iteration for \( i = 1, 2, \cdots, n \), we have that
\[ f_n = h_n + \sum_{i=1}^{n} h_{i-1}G_i \prod_{k=i+1}^{n} (1 + G_k) + \sum_{i=1}^{n} c_i \prod_{k=i+1}^{n} (1 + G_k). \]

Hence,
\[ |I(a, x) - \{ f(x) - h(x) \}| \]
\[ = \left| h(x) + I(a, x) - h_n \right| \]
\[ \leq \left| \sum_{i=1}^{n} h_{i-1}G_i \prod_{k=i+1}^{n} (1 + G_k) - \sum_{i=1}^{n} c_i \prod_{k=i+1}^{n} (1 + G_k) \right| \]
\[ + \left| I(a, x) - \sum_{i=1}^{n} h_{i-1}G_i \prod_{k=i+1}^{n} (1 + G_k) \right| + \left| \sum_{i=1}^{n} [-c_i] \prod_{k=i+1}^{n} (1 + G_k) \right| \]
\[ < \sum_{i=1}^{n} |h_{i-1}G_i| \prod_{k=i+1}^{x} (1 + G_k) + \sum_{k=i+1}^{n} (1 + G_k) + \epsilon/3 + B[\epsilon(3B)^{-1}] \]
\[ < B[\epsilon(3B)^{-1}] + 2\epsilon/3 = \epsilon. \]

Therefore, (1) implies (2).

Proof. (2) \Rightarrow (1). Since \( h \) is bounded on \([a, b]\) and \( G \in OB^0\) on \([a, b]\), it follows that \( f \) is bounded on \([a, b]\). Since \( G \) is in \( OM^0 \) and \( OB^0 \) on \([a, b]\), it follows from Lemma 1 that \( G \in OA^0 \) on \([a, b]\). Further, it follows by employing Lemma 2 that \( f(u)G(u, v) \in OA^0 \) on \([a, b]\). Suppose \( a \leq x \leq b \). If \( a = x \), then (2) follows immediately. Therefore, suppose \( a < x \). Let \( \epsilon > 0 \).

There exists a subdivision \( D_1 \) of \([a, x]\) such that, if \( \{x_i\}_{i=0}^{n} \) is a refinement of \( D_1 \), then
\[ \left| \int_{a}^{x} f(u)G(u, v) - \sum_{i=1}^{n} f_{i-1}G_i \right| < \frac{\epsilon}{3}. \]
There exists a number \( B \) such that, if \( a \leq t \leq x \), then \( |I(a, t)| < B \), where \( I(a, t) \) is defined as in the first part of the proof. Now, since \( G \in OM^0 \) on \([a, x]\), there exists a subdivision \( D_2 \) of \([a, x]\) such that, if \( \{x_i\}_{i=0}^n \) is a refinement of \( D_2 \), then

\[
\sum_{i=1}^{n} \left| 1 + G_i - \prod_{i=1}^{x_i} (1 + G) \right| < \epsilon (3B)^{-1}.
\]

It follows by applying Lemma 3 that there exists a subdivision \( D_3 \) of \([a, x]\) such that, if \( \{x_i\}_{i=0}^n \) is a refinement of \( D_3 \), then

\[
\sum_{i=1}^{n} |h_{i-1} G_i - I(x_{i-1}, x_i)| < \frac{\epsilon}{3}.
\]

Let \( D \) denote the subdivision \( \bigcup_{i=1}^{3} D_i \) of \([a, x]\). Suppose \( \{x_i\}_{i=0}^n \) is a refinement of \( D \). Thus,

\[
\left| h(x) + \int_{a}^{x} f(u) G(u, v) - f(x) \right|
\]

\[
\leq \left| h(x) + \sum_{i=1}^{n} f_{i-1} G_i - h(x) - I(a, x) \right| + \left| \int_{a}^{x} f(u) G(u, v) - \sum_{i=1}^{n} f_i G_i \right|
\]

\[
< \left| \sum_{i=1}^{n} \{h_{i-1} + I(a, x_{i-1})\} G_i - I(a, x) \right| + \epsilon/3
\]

\[
= \left| \sum_{i=1}^{n} \{h_{i-1} G_i + [I(a, x_{i-1})][1 + G_i] - I(a, x_{i-1})\} - I(a, x) \right| + \epsilon/3
\]

\[
< \left| \sum_{i=1}^{n} \{I(a, x_{i-1})\} \left[ x_{i-1} \prod_{i=1}^{x_{i-1}} (1 + G) \right] - I(a, x_{i-1}) \right| + \epsilon/3
\]

\[
+ \sum_{i=1}^{n} |I(a, x_{i-1})| \left| 1 + G_i - \prod_{i=1}^{x_i} (1 + G) \right| + \epsilon/3
\]

\[
< \left| \sum_{i=1}^{n} \{I(a, x_{i-1}) - I(a, x_{i-1})\} - I(a, x) \right| + \epsilon/3
\]

\[
+ \sum_{i=1}^{n} |h_{i-1} G_i - I(x_{i-1}, x_i)| + B(\epsilon (3B)^{-1}) + \epsilon/3
\]

\[
< 0 + \epsilon/3 + 2\epsilon/3 = \epsilon.
\]

Therefore, (2) implies (1).
Theorem 2. If $f$ and $h$ are functions from $R$ to $N$, $G$ is a function from $R \times R$ to $N$, \( G(y, x) = G(x, y) \) for $a \leq x < y \leq b$, $h$ is quasi-continuous on $[a, b]$ and $G \in OB^o$ on $[a, b]$, then the following statements are equivalent:

1. $f$ is bounded on $[a, b]$, $G \in OA^o$ on $[a, b]$, $G(u, v)f(u) \in OA^o$ on $[a, b]$ and $f(x) = h(x) + \int_a^x G(u, v)f(u) \, du$ for $a \leq x \leq b$, and

2. $G \in OM^o$ on $[b, a]$ and $\int_a^x (1 + G)(u, v)h(u) \, du$ exists and is $f(x) - h(x)$ for $a \leq x \leq b$.

The proof of Theorem 2 is similar to the proof of Theorem 1, and therefore, we omit it.

REFERENCES