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THE NUMBER OF MULTIPLICATIONS ON H-SPACES OF TYPE (3, 7)

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ABSTRACT. The technique of homotopy localization is used to give an enumeration of the multiplications on H-spaces of type (3, 7).

1. **Introduction.** Let $S^3 \to \mathrm{Sp}(2) \to S^7$ be the standard fibration with characteristic element $\omega \in \pi_7(B_{S^3})$ and let $n: S^7 \to S^7$ be the map of degree n. Denote by $S^3 \to X_n \to S^7$ the fibration induced by $n\omega = \omega \circ n$ in $\pi_7(B_{S^3})$. We then have the induced fibre square

$$\begin{array}{ccc} X_n & \xrightarrow{\tilde{n}} & \mathrm{Sp}(2) \\ \downarrow & & \downarrow \\ S^7 & \xrightarrow{n} & S^7 \end{array}$$

where $\tilde{n}: X_n \to \mathrm{Sp}(2)$ is a map covering n.

Since $\pi_7(B_{S^3}) = \pi_6(S^3) = Z/12Z$ and ω is a generator of this group, we obtain seven distinct homotopy types for X_n: $X_0 = S^3 \times S^7$, $X_1 = \mathrm{Sp}(2)$, X_2, X_3, X_4, X_5 (the Hilton-Roitberg H-space) and X_6. It is known that all of these except X_2 and X_6 are H-spaces [6], [9] and that X_2 and X_6 are not H-spaces [12, I]. Furthermore, it is also known (see [7] and [12, II]) that these five H-spaces are all the simply-connected H-spaces of type (3, 7).

In this note we formulate a general method of calculating the number (of homotopy classes) of multiplications on an H-space by means of localization. We apply it to the above situation and prove

Theorem 1.1. *Let $\#(X_n)$ denote the number of multiplications of X_n. Then*

(a) $\#(X_0) = 2^{38} \cdot 3^{15} \cdot 5^5 \cdot 7$,

(b) $\#(X_1) = 2^{20} \cdot 3 \cdot 5^5 \cdot 7$,

(c) $\#(X_3) = 2^{20} \cdot 3^{15} \cdot 5^5 \cdot 7$,

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(d) $\#(X_4) = 2^{38} \cdot 3 \cdot 5^5 \cdot 7$,
(e) $\#(X_5) = 2^{20} \cdot 3 \cdot 5^5 \cdot 7$.

Of course (a) gives the number of multiplications on $S^3 \times S^7$ which is an easy consequence of a result of Copeland [3]. Also (b) gives the number of multiplications on $\mathrm{Sp}(2)$ as calculated by Mimura [8]. We shall use the results of Copeland and Mimura to obtain (c), (d) and (e).

In §2 we present the general constructions which we will use and in the final section we do the actual computations.

2. **Localization.** Although it is possible to do localization on more general spaces, for convenience we work in the category of 1-connected CW-spaces. We give a brief summary of localization in homotopy theory following Sullivan [10, §2].

Let l be a subset of the prime integers (we do not exclude the case $l = \emptyset$, the empty set). By Z_l is meant the set of rational numbers with denominators prime to the primes in l. We say that Y is an *l-local space* if the homotopy groups $\pi_*(Y)$ are all Z_l-modules.

A map $u\colon X \to X_l$, where X_l is an l-local space, is called a *localization* (with respect to l) if for any $f\colon X \to Y$ with Y an l-local space, there is a map $f'\colon X_l \to Y$, unique up to homotopy, such that $f' \circ u = f$. Thus $u^*\colon [X_l, Y] \to [X, Y]$ is a bijection. By [10, p. 2.2], $u\colon X \to X_l$ is a localization if and only if u *localizes homotopy*, i.e., there is an isomorphism $\pi_*(X_l) \approx \pi_*(X) \otimes Z_l$ such that the following diagram commutes:

$$\begin{tikzcd} \pi_*(X) \arrow[r, "u_*"] \arrow[dr, "\theta"'] & \pi_*(X_l) \arrow[d, phantom, "\wr\wr"] \\ & \pi_*(X) \otimes Z_l \end{tikzcd}$$

where $\theta(x) = x \otimes 1$. Using homology groups instead of homotopy groups, we similarly define "u localizes homology" and observe that this is equivalent to u being a localization (since our spaces are 1-connected).

It is also known [10, p. 2.9] that for a space X and any set of primes l, a localization $u\colon X \to X_l$ always exists.

We shall need a result about localization and homotopy sets which we state as Proposition 2.1.

Note that if $m \subset l$ then there is a unique map $X_l \to X_m$ which is compatible with localizations. These maps appear in the following proposition.

Proposition 2.1. *If l and k are sets of primes and A is a finite CW-complex then the following is a pull-back diagram in the category of sets:*

$$\begin{array}{ccc} [A, X_{l\cup k}] & \longrightarrow & [A, X_l] \\ \downarrow & & \downarrow \\ [A, X_k] & \longrightarrow & [A, X_{l\cap k}] \end{array}$$

For a proof see [5] or [1, p. 147].

Theorem 2.2. *Let X be an H-space which is a finite CW-complex such that $[X \wedge X, X_\phi] = 0$.*[2] *Let l be a set of primes and let $\bar{l}$ be the complementary set of primes. Assume that there are spaces R and S such that $X_l \simeq R_l$ and $X_{\bar{l}} \simeq S_{\bar{l}}$. Then R_l and $S_{\bar{l}}$ are H-spaces and $\#(X) = \#(R_l) \cdot \#(S_{\bar{l}})$.*

Proof. Since X_l and $X_{\bar{l}}$ are H-spaces it follows immediately that R_l and $S_{\bar{l}}$ are H-spaces.

By [2], $\#(Y) = \#([Y \wedge Y, Y])$ for any H-space Y, where we ambiguously use the symbol $\#$ to denote both the cardinality of a set and the number of multiplications on an H-space. By 2.1 and the hypothesis that $[X \wedge X, X_\phi] = 0$, we have that

$$\#(X) = \#([X \wedge X, X]) = \#([X \wedge X, X_l]) \cdot \#([X \wedge X, X_{\bar{l}}]).$$

But $u \wedge u: X \wedge X \to X_l \wedge X_l$ clearly localizes homology and thus is a localization. Therefore, by the definition of localization,

$$\#([X \wedge X, X_l]) = \#([X_l \wedge X_l, X_l]) = \#([R_l \wedge R_l, R_l]) = \#(R_l).$$

Similarly $\#([X \wedge X, X_{\bar{l}}]) = \#(S_{\bar{l}})$ and so

$$\#(X) = \#(R_l) \cdot \#(S_{\bar{l}}).$$

Corollary 2.3. (a) $\#(X_3) = \#(\mathrm{Sp}(2)_{\bar{3}}) \cdot \#((S^3 \times S^7)_3)$,
(b) $\#(X_4) = \#(\mathrm{Sp}(2)_{\bar{2}}) \cdot \#((S^3 \times S^7)_2)$,
(c) $\#(X_5) = \#(\mathrm{Sp}(2))$.

Proof. We just prove (a) since the proofs of (b) and (c) are similar. Clearly $(X_3)_\phi = K(Q, 3) \times K(Q, 7)$ and so $[X_3 \wedge X_3, (X_3)_\phi] = H^3(X_3 \wedge X_3; Q) \oplus H^7(X_3 \Delta X_3; Q) = 0$ for dimensional reasons. From §1 we have induced fibre squares

[2] This condition is equivalent to there being a finite number of multiplications on X.

$$\begin{array}{ccccc} S^3 \times S^7 & \xrightarrow{\tilde{4}} & X_3 & \xrightarrow{\tilde{3}} & \mathrm{Sp}(2) \\ \downarrow & & \downarrow & & \downarrow \\ S^7 & \xrightarrow{4} & S^7 & \xrightarrow{3} & S^7 \end{array}$$

with identity maps on the fibres S^3. Since $3_* \otimes 1: \pi_*(S^7) \otimes Z_{\bar{3}} \to \pi_*(S^7) \otimes Z_{\bar{3}}$ is an isomorphism, it follows from the five lemma and the exact homotopy sequence of a fibration that

$$\tilde{3}_* \otimes 1: \pi_*(X_3) \otimes Z_{\bar{3}} \to \pi_*(\mathrm{Sp}(2)) \otimes Z_{\bar{3}}$$

is an isomorphism. Hence $(X_3)_{\bar{3}} \simeq (\mathrm{Sp}(2))_{\bar{3}}$. Similarly $(X_3)_{\bar{2}} \simeq (S^3 \times S^7)_{\bar{2}}$ and, in particular, $(X_3)_3 \simeq (S^3 \times S^7)_3$. The result now follows by Theorem 2.3.

3. **Computations.** By §2 all we must do is evaluate the numbers $\#(\mathrm{Sp}(2)_{\bar{3}})$, $\#(\mathrm{Sp}(2)_{\bar{2}})$, $\#((S^3 \times S^7)_3)$ and $\#((S^3 \times S^7)_2)$.

Mimura [8] has calculated $\#(\mathrm{Sp}(2))$ by showing the order of the finite nilpotent group $[\mathrm{Sp}(2) \wedge \mathrm{Sp}(2), \mathrm{Sp}(2)]$ to be $2^{20} \cdot 3 \cdot 5^5 \cdot 7$. By applying (algebraic) localization to nilpotent groups [4], it is easily seen that the order of the l-localization of a nilpotent group of order n is just the product of those prime power factors of n with the primes in l. Thus, since $[\mathrm{Sp}(2)_l \wedge \mathrm{Sp}(2)_l, \mathrm{Sp}(2)_l] = [\mathrm{Sp}(2) \wedge \mathrm{Sp}(2), \mathrm{Sp}(2)_l]$ is just the l-localization of the finite nilpotent group $[\mathrm{Sp}(2) \wedge \mathrm{Sp}(2), \mathrm{Sp}(2)]$, it follows that

$$\#(\mathrm{Sp}(2)_{\bar{3}}) = 2^{20} \cdot 5^5 \cdot 7 \quad \text{and} \quad \#(\mathrm{Sp}(2)_{\bar{2}}) = 3 \cdot 5^5 \cdot 7.$$

To obtain $\#((S^3 \times S^7)_l)$ we employ a mild generalization of the work of Copeland [3] together with the fact that $\pi_*(X_l) \approx \pi_*(X) \otimes Z_l$. This produces the following results for $i = 2, 3$:

$$\#((S^3 \times S^7)_i) = \prod_{j=3,7} (a_{i,j} \cdot b_{i,j}^2 \cdot c_{i,j} \cdot d_{i,j}^2 \cdot e_{i,j}^2 \cdot f_{i,j})$$

where

$$a_{i,j} = \#([S^3 \wedge S^7 \wedge S^3 \wedge S^7, (S^j)_i]) = \#(\pi_{20}(S^j) \otimes Z_i),$$

$$b_{i,j} = \#([S^3 \wedge S^7 \wedge S^7, (S^j)_i]) = \#(\pi_{17}(S^j) \otimes Z_i),$$

$$c_{i,j} = \#([S^7 \wedge S^7, (S^j)_i]) = \#(\pi_{14}(S^j) \otimes Z_i),$$

$$d_{i,j} = \#([S^3 \wedge S^3 \wedge S^7, (S^j)_i]) = \#(\pi_{13}(S^j) \otimes Z_i),$$

$$e_{i,j} = \#([S^3 \wedge S^7, (S^j)_i]) = \#(\pi_{10}(S^j) \otimes Z_i),$$

$$f_{i,j} = \#([S^3 \wedge S^3, (S^j)_i]) = \#(\pi_6(S^j) \otimes Z_i).$$

By using Toda's results [11] on the homotopy groups of S^3 and S^7, we easily obtain $\#((S^3 \times S^7)_2) = 2^{38}$ and $\#((S^3 \times S^7)_3) = 3^{15}$.

This completes the proof of Theorem 1.1.

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