A FIXED POINT CRITERION FOR LINEAR REDUCTIVITY

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ABSTRACT. Let \( G \) be a linear algebraic group over an algebraically closed field. If for all actions of \( G \) on smooth schemes, the fixed point scheme is smooth, then \( G \) is linearly reductive under either of the additional assumptions: (a) the ground field is characteristic zero, or (b) \( G \) is connected, reduced, and solvable.

Let \( K \) be an algebraically closed field and \( G \) a linear algebraic group over \( K \). Say \( G \) has the smooth fixed point property if for all actions of \( G \) on a smooth scheme, the fixed point scheme is also smooth. Fogarty [1] has shown that if \( G \) is linearly reductive, then \( G \) has the smooth fixed point property. One can ask the converse question: If \( G \) is not linearly reductive, is there an action of \( G \) on a smooth scheme with a nonsmooth fixed point scheme? In this note we show how to construct such an action for any \( G \) that is a split extension by a unipotent subgroup. This gives an affirmative answer to the question for any class of groups where \( G \) being not linearly reductive implies \( G \) is a split extension by a unipotent subgroup. In particular this includes all groups of characteristic zero and in arbitrary characteristic connected reduced solvable groups.

Let \( G \) be a linear algebraic group over \( K \) which is a split extension by the unipotent subgroup \( U \). We first show that we may assume that \( U \) is the direct sum of copies of the additive group. If \( G \) modulo a normal subgroup has an action with a nonsmooth fixed point scheme, then certainly \( G \) does. Using this we can replace \( G \) by \( G \) modulo the commutator subgroup of \( U \), and hence we may assume \( G \) is commutative. In characteristic zero this already implies \( U \) is the direct sum of copies of the additive group. In characteristic \( p \) there are truncated Witt groups; however, if we take \( G \) modulo \( p \cdot U \) we may assume \( G \) is the direct sum of additive groups by a theorem of Serre [2].

We now construct an action of \( G \) on affine space with a nonreduced fixed point scheme. Using our assumption that \( G \) is a split extension, we
pick a section for the exact sequence

\[ 0 \rightarrow U \rightarrow G \rightarrow G/U \rightarrow 0. \]

Using this section we write elements of \( G \) as ordered pairs \((u, t)\) with \( u \in U \) and \( t \in G/U \). Let \( \rho: G/U \rightarrow \text{Aut}(U) \) give the action of \( G/U \) on \( U \).

The multiplication in \( G \) is given by

\[ (u, t) \cdot (u', t') = (u + \rho(t) \cdot u', t \cdot t'). \]

Let \( X = U \times \text{spec } K[x] = \text{spec } K[y_1, \cdots, y_n, x] \). In terms of these coordinates on \( U \), let \( \rho(t) \) be given by the matrix \((\rho_{ij}(t))\). The action of \( G \) on \( X \) is given by

\[
(u, t): \begin{cases}
x \mapsto x, \\
y_i \mapsto \sum_j \rho_{ij}(t)y_j + u_ix^2, \quad i = 1, \cdots, n,
\end{cases}
\]

where \( u = (u_1, u_2, \cdots, u_n) \). We now verify that this is an action. Let \((u', t') = (u'_1, u'_2, \cdots, u'_n, t)\) be another element of \( G \). Now \( x \) is fixed so there is nothing to do with \( x \).

\[
(u, t): y_i \mapsto \sum_j \rho_{ij}(t)y_j + u_ix^2,
\]

\[
(u', t'): \sum \rho_{ij}(t)y_j + u_ix^2 \mapsto \sum \rho_{ij}(t) \left( \sum k \rho_{jk}(t')y_k + u'_{j}x^2 \right) + u_ix^2 \sum k \rho_{ik}(t \cdot t')y_k + \left( \sum j \rho_{ij}(t) \cdot u'_j + u_i \right) \cdot x^2.
\]

But this is also the result of \((u + \rho(t) \cdot u', t \cdot t') = (u, t) \cdot (u', t')\) acting on \((x, y_1, \cdots, y_n)\).

Now \( X \) is affine \( n + 1 \) space and hence smooth. On the other hand, the fixed point scheme of the action by \( G \) is defined by the ideal \( I \) generated by all the elements of the form \( gr - r \) for \( r \in K[y_1, \cdots, y_n, x] \) and \( g \in G \). By setting \( t \) to be the identity and \( u_1 = 1 \) we see that \( x^2 \in I \). But it is clear that \( x \not\in I \) and so the fixed point scheme is not reduced.

**BIBLIOGRAPHY**