ON A NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

NGUYEN PHUONG CAC

ABSTRACT. Consider a bounded domain $G \subset \mathbb{R}^N (N \geq 1)$ with smooth boundary $\Gamma$. Let $L$ be a uniformly elliptic linear differential operator. Let $\gamma$ and $\beta$ be two maximal monotone mappings in $\mathbb{R}$. We prove that, when $\gamma$ satisfies a certain growth condition, given $f \in L^2(G)$ there is $u \in H^2(G)$ such that

$$Lu + \gamma(u) = f \text{ a.e. on } G, \quad -\frac{\partial u}{\partial \nu} \in \beta(u) \text{ a.e. on } \Gamma,$$

where $\frac{\partial u}{\partial \nu}$ is the conormal derivative associated with $L$.

1. Let $G \subset \mathbb{R}^N (N \geq 1)$ be a bounded domain with smooth boundary $\Gamma$. Consider the uniformly elliptic linear operator

$$Lu = -\sum_{i,j} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(x) \frac{\partial u}{\partial x_i} + c(x) u,$$

$$a_{ij} = a_{ij} \in C^1(\overline{G}); \quad b_i, c \in L^\infty(G), \quad (i, j = 1, 2, \ldots, N),$$

$$a_{ij}(x) \xi_i \xi_j \geq c|\xi|^2, \quad c > 0 \text{ constant}, \forall x \in G, \xi \in \mathbb{R}^N.$$

(All functions and scalars that we consider are real.)

Let $\gamma : \mathbb{R} \to 2^{\mathbb{R}}$ be a maximal monotone mapping. The domain $D(\gamma)$ of $\gamma$ is the set of all numbers $s$ such that $\gamma(s) \neq \emptyset$. For each $s \in D(\gamma)$, $\gamma(s)$ is a closed interval and thus contains a unique element, which we denote by $\gamma^0(s)$, having smallest absolute value. We assume that the mapping $\gamma$ satisfies the condition

$$|\gamma^0(s)| \geq \phi(s)|s|, \quad \forall s \in D(\gamma) \text{ with } \lim_{|s| \to \infty} \phi(s) = \infty. \tag{1}$$

It can be verified that $\gamma$ induces a maximal monotone mapping $\overline{\gamma} : L^2(G) \to 2L^2(G)$ in a natural way:

$$\overline{\gamma}(u) = \{v \in L^2(G) | v(x) \in \gamma(u(x)) \text{ a.e.} \} \quad (u \in L^2(G)).$$
Similarly, let $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ be another maximal monotone mapping and let $\overline{\beta}$ be the mapping in $L^2(\Gamma)$ induced by $\beta$.

**Proposition.** Suppose that all the conditions on the operator $L$ and the mappings $\gamma$ and $\beta$ described above are satisfied. Suppose also that there exists $s_0 \in D(\gamma) \cap D(\beta)$ with $0 \in \beta(s_0)$. Then for every given $f \in L^2(G)$, there exists $u \in H^2(G)$ such that

(2) $Lu + \overline{\gamma}(u) \ni f$ in the sense of $L^2(G),$

(3) $-\partial u / \partial v = a_{ij} D_i u \cdot \cos(n, x_i) \in \overline{\beta}(u|_{\Gamma}),$

where $n$ is the outward normal to $\Gamma$.

Before proving this proposition in §2, we would like to make a few comments. When (i) $\gamma \equiv 0$ and the bilinear form

$$a(u, v) = \int_G (a_{ij} D_i u D_j v + b_{ij} D_i u v + c u v) \, dx$$

is coercive on $H^1(G)$, or (ii) $Lu = -\Delta u + u$, the proposition has been proved by H. Brezis in [3] and [2] respectively. It seems to us that the general case, where no coercivity is assumed, cannot be immediately reduced to these cases.

The corresponding Dirichlet problem

(4) $Lu + \overline{\gamma}(u) \ni f,$

(5) $u|_{\Gamma} = 0,$

has been studied by P. Hess [7], and condition (1) on $\gamma$ is similar to his condition in [7]. Our method can also be applied to this Dirichlet problem, and the argument will be simpler than the Neumann-type problem considered here. We feel that the main difference between our method as applied to the Dirichlet problem and that of P. Hess lies in proving the existence of a solution for the approximate equation. He uses a theorem on the solvability of a functional equation involving a demicontinuous mapping of type $(S^+)$ which is similar to one established by him earlier [6, Theorem 1], using a homotopy argument. Instead, we shall use the well-known Schauder fixed point theorem (see e.g. [4, p. 105]).

2. The proof makes use of the concept of the Yosida approximation. Let $U : H \to 2^H$ be a maximal monotone operator in a Hilbert space $H$. Then for every $\epsilon > 0$, $(I + \epsilon U)^{-1}$ is a nonexpansive mapping defined on all of $H$. The mapping

(6) $U_\epsilon = [I - (I + \epsilon U)^{-1}] / \epsilon$
is called the Yosida approximation of $U$ (at $\epsilon$). $U_\epsilon$ is Lipschitzian with Lipschitz constant $2/\epsilon$ and monotone. For more details on Yosida approximations we refer the reader to M. Crandall and A. Pazy [5], or T. Kato [8]. In the case of the maximal monotone mappings $\gamma$ and $\beta$ introduced in \textsection 1, it can be verified that the Yosida approximation of $\bar{\gamma}$ for example is generated by $\gamma_\epsilon$.

Proof of the proposition. We observe that by shifting and changing variable, we can assume without loss of generality that $0 \leq \gamma(0)$ and $0 \leq \beta(0)$. Then $\gamma_\epsilon(0) = \beta_\epsilon(0) = 0$. The proof consists of proving that the approximate problem

$$(7) \quad Lu + \gamma_\epsilon(u) = f,$$

$$(8) \quad -\partial u/\partial v = \beta_\epsilon(u|_{\Gamma})$$

has a solution $u_\epsilon \in H^2(G)$ for all $\epsilon > 0$ sufficiently small. We then pass to the limit as $\epsilon \downarrow 0$ using estimates for $u_\epsilon$ independent of $\epsilon$.

1. Proof that the approximate problem has a solution. Let

$$L'u = -D_j(a_{ij}D_i u), \quad L'' = L - L',$$

$$a'(u, v) = \int_G a_{ij}D_i uD_j v \, dx \quad (u, v \in H^1(G)).$$

For $u$ given in $H^1(G)$, the linear form

$$w \rightarrow a'(u, w) + \int_G [\gamma_\epsilon(u) + u] w \, dx + \int_\Gamma \beta_\epsilon(u|_{\Gamma}) w \, d\Gamma \quad (w \in H^1(G)),$$

is continuous on $H^1(G)$, so that there is an element $\mathcal{A}u \in [H^1(G)]'$ with

$$\langle \mathcal{A}u, w \rangle = a'(u, w) + \int_G [\gamma_\epsilon(u) + u] w \, dx + \int_\Gamma \beta_\epsilon(u|_{\Gamma}) w \, d\Gamma \quad (w \in H^1(G)).$$

It can be verified that the mapping $u \rightarrow \mathcal{A}u$ is bounded, hemicontinuous, strictly monotone and coercive. Therefore (see [4, Theorem 1] or [9, Chapter 2, Theorem 2.1]) for every given $v \in H^1(G)$ there exists a unique $u \in H^1(G)$ such that for all $w \in H^1(G)$

$$\langle \mathcal{A}u, w \rangle = a'(u, w) + \int_G [\gamma_\epsilon(u) + u] w \, dx + \int_\Gamma \beta_\epsilon(u|_{\Gamma}) w \, d\Gamma$$

$$= \int_G (f - L''v + v) w \, dx.$$

We then deduce that the boundary value problem

$$(9) \quad L'u + \gamma_\epsilon(u) + u = f - L''v + v \quad \text{in the sense of } \mathcal{D}'(G),$$

$$(10) \quad -\partial u/\partial v = \beta_\epsilon(u|_{\Gamma})$$

has a unique solution $u_\epsilon \in H^1(G)$. If we bring $\gamma_\epsilon(u)$ to the right-hand side in equation (9), then it follows from [3, Theorem I. 10], that $u_\epsilon \in H^2(G)$. 

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It now suffices to show that the mapping $T_\epsilon: v \to u_\epsilon$ in $H^1(G)$ has a fixed point.

(a) The mapping $T_\epsilon$ is continuous. Let $v_1, v_2 \in H^1(G)$ and $u_1 = T_\epsilon(v_1), u_2 = T_\epsilon(v_2)$. The continuity of $T_\epsilon$ can be seen by taking the difference of equations (9) corresponding to $v_1$ and $v_2$ and then taking the inner product in $L^2(G)$ of this with $u_1 - u_2$.

(b) There is an integer $K > 0$ and an $\epsilon_0 > 0$ such that $T_\epsilon(0 < \epsilon < \epsilon_0)$ maps the closed ball $B_K(0)$ of $H^1(G)$ into itself. If not, for each $\gamma = 1, 2, \ldots$ there are $\gamma_n$ with $0 < \gamma_n < n^{-1}$, $u_n$ with $\|u_n\|_1 \leq n$, $\gamma_n$ with $\|\gamma_n\| > n$ (where $\|\cdot\|_1$ is the norm in $H^1(G)$) satisfying (9) and (10) with $\epsilon = \gamma_n$. Taking the inner product in $L^2(G)$ of (9) with $n^{-2}\gamma_n u_n$ we obtain
\[
\int_G n^{-2} \beta_{\gamma_n}(u_n) u_n \, d\gamma + \int_G n^{-2} \gamma_{\gamma_n}(u_n) u_n \, dx + \|n^{-2} u_n\|_0^2 = \|n^{-1}(f - L''u_n + v_n)\|_0 \cdot \|n^{-2} u_n\|_0,
\]
where $\|\cdot\|_0$ is the norm in $L^2(G)$. From this we deduce
\[
\begin{align*}
1 &< \|n^{-2} u_n\|_1 < C \quad (n = 1, 2, \ldots), \\
\int_G n^{-2} \gamma_{\gamma_n}(u_n) u_n \, dx &< C \quad (n = 1, 2, \ldots).
\end{align*}
\]

Here and in the sequel $C$ denotes various positive constants independent of $\gamma, \beta, \epsilon$. Now taking the inner product in $L^2(G)$ of (9) with $n^{-2}\gamma_{\gamma_n}(u_n)$ we obtain (for justification see [1] or [10, Appendix I] for a special case), recalling $w_n = n^{-1} u_n$:
\[
\begin{align*}
\int_G \gamma'_{\gamma_n}(u_n) a_{ij} D_i D_j w_n \, dx + \int_G n^{-2} \beta_{\gamma_n}(u_n) \gamma_{\gamma_n}(u_n) d\gamma + \int_G n^{-2} \gamma_{\gamma_n}(u_n) u_n \, dx \\
\quad + \|n^{-1} \gamma_{\gamma_n}(u_n)\|_0^2 \leq \|n^{-1}(f - L''u_n + v_n)\|_0 \cdot \|n^{-1} \gamma_{\gamma_n}(u_n)\|_0.
\end{align*}
\]
The third integral is nonnegative. Since $0 \in \gamma(0)$, $\gamma_{\gamma_n}(\cdot)$ is monotone increasing so that $\gamma'_{\gamma_n}(u_n(x)) \geq 0$ a.e. and the first integral is also nonnegative. Moreover, we observe that if $u_n(x) = 0$ then $\beta_{\gamma_n}(u_n) \gamma_{\gamma_n}(u_n) = 0$, and if $u_n(x) \neq 0$ then
\[
\beta_{\gamma_n}(u_n) \gamma_{\gamma_n}(u_n) = \beta_{\gamma_n}(u_n) u_n \cdot \gamma_{\gamma_n}(u_n) u_n \cdot u_n^2 \geq 0,
\]
so that the second integral is also nonnegative. We then deduce that
\[
\|n^{-1} \gamma_{\gamma_n}(u_n)\|_0 < C \quad (n = 1, 2, \ldots).
\]

We now write
(13) \[ L'w_n + w_n = n^{-1}(f - L''v_n + v_n - \gamma \epsilon_n(u_n)), \]

(14) \[ -\partial w_n / \partial \nu = n^{-1}\beta \epsilon_n(u_n|\mathbf{r}). \]

Since the right-hand side of (13) remains bounded in \( L^2(G) \), it follows from [3, Theorem 1.10], that \( \|w_n\|_2 < C (n = 1, 2, \ldots) \), where \( \|\cdot\|_2 \) denotes the norm in \( H^2(G) \). Because the imbedding of \( H^2(G) \) into \( H^1(G) \) is compact, we can extract a subsequence, still denoted by \( \{w_n\} \), such that \( w_n \) converges strongly in \( H^1(G) \) to \( w \) and \( w_n \) converges a.e. on \( G \) to \( w \). Since \( \|w\|_1 \geq 1 (n = 1, 2, \ldots) \), \( w(x) \neq 0 \) on a subset of \( G \) of nonzero measure. We shall see that this contradicts condition (1) on \( \gamma \) and (12). In fact, putting \( s_n(x) = (1 + \epsilon_n \gamma)^{-1}u_n(x) \), we obtain with \( t_n(x) \in \gamma(s_n(x)) \)

(15) \[ u_n(x) = s_n(x) + \epsilon_n t_n(x), \]

(16) \[ n^{-2} \gamma \epsilon_n(u_n(x))u_n(x) = n^{-2} u_n(x) t_n(x) = |u_n(x)| \cdot n^{-1}|t_n(x)|. \]

Consider \( x \in G \) with \( \lim n \|w_n(x)\| > 0 \), i.e. \( \lim n \|u_n(x)\| = \infty \). Then

\[ \lim n^{-1}|t_n(x)| = \infty. \]

For otherwise there would be a subsequence such that

\[ \sup_k n_k^{-1}|t_{n_k}(x)| < \infty. \]

From (15) it then follows that

(17) \[ \lim \inf_k n_k^{-1}|s_{n_k}(x)| \geq \lim_k n_k^{-1}|u_{n_k}(x)| > 0. \]

By condition (1) on \( \gamma \),

\[ n_k^{-1}|t_{n_k}(x)| \geq n_k^{-1}|\gamma \phi(s_{n_k}(x))| > \phi(s_{n_k}(x)) n_k^{-1}|s_{n_k}(x)|. \]

Since \( \lim_k |s_{n_k}(x)| = \infty \), \( \lim_k \phi(s_{n_k}(x)) = \infty \). This together with (17) shows that

\[ \lim_k n_k^{-1}|t_{n_k}(x)| = \infty \]

and we thus arrive at a contradiction. From (16) we therefore see that

\[ \lim n^{-2} \gamma \epsilon_n(u_n(x))u_n(x) = \infty \]

on a subset of \( G \) of nonzero measure. By Fatou's lemma, this contradicts (12).

(c) The mapping \( T_\epsilon (0 < \epsilon < \epsilon_0) \) of \( B_K(0) \) into itself is relatively compact. In fact, by an argument similar to that in the last step, we see
that for all \( v \in B_K^1(0) \), \( \| T_\epsilon(v) \|_2 < C \). Since the imbedding of \( H^2(G) \) into \( H^1(G) \) is compact, we deduce that the closure of \( T_\epsilon(B_K^1(0)) \) is compact.

Thus by the Schauder fixed point theorem \([4, \text{p. 105}]\), \( T_\epsilon(0 < \epsilon < \epsilon_0) \) has a fixed point in \( B_K(0) \).

II. \textit{Passing to the limit as} \( \epsilon \downarrow 0 \). Using the same argument as in Step I(b) above (take the inner product in \( L^2(G) \) of (7) with \( u \) and then with \( \tilde{\gamma}_\epsilon(u) \)), we see that there is a constant \( C \) independent of \( \epsilon \) such that a solution \( u_\epsilon \) of (7) and (8) satisfies

\[
\| u_\epsilon \|_2 < C, \quad \| \tilde{\gamma}_\epsilon(u_\epsilon) \|_0 < C \quad (0 < \epsilon < \epsilon_0).
\]

Since the mapping \( u \rightarrow u \mid_\Gamma \) of \( H^1(G) \) onto \( H^{1/2}(\Gamma) \subset L^2(\Gamma) \) is continuous, we can extract a subsequence \( \{ u_{\epsilon_n} \} \) with the following properties

- \( u_{\epsilon_n} \) converges weakly to \( u \) in \( H^2(G) \),
- \( L u_{\epsilon_n} \) converges weakly to \( L u \) in \( L^2(G) \),
- \( u_{\epsilon_n} \) converges strongly to \( u \) in \( H^1(G) \),
- \( \tilde{\gamma}_{\epsilon_n}(u_{\epsilon_n}) \) converges weakly to \( -Lu + f \) in \( L^2(G) \),
- \( \partial u_{\epsilon_n} / \partial \nu \) converges weakly to \( \partial u / \partial \nu \) in \( L^2(\Gamma) \)

(i.e. \( \tilde{\beta}_{\epsilon_n}(u_{\epsilon_n}) \) converges weakly to \( -\partial u / \partial \nu \) in \( L^2(\Gamma) \)),

- \( u_{\epsilon_n} \mid_\Gamma \) converges strongly to \( u \mid_\Gamma \) in \( L^2(\Gamma) \).

From a property of Yosida approximations \([8, \text{Lemma 4.5}]\), it then follows that

\[
u \in D(\tilde{\gamma}), \quad -Lu + f \in \tilde{\gamma}(u); \quad u \mid_\Gamma \in D(\tilde{\beta}), \quad -\partial u / \partial \nu \in \tilde{\beta}(u \mid_\Gamma)
\]

and the proof is complete.

From the proposition we deduce the following

**Corollary.** \textit{Suppose that the conditions in the proposition are satisfied. Then for any} \( k_1 \geq 0, k_2 > 0 \), \textit{the boundary value problem}

\[
Lu + \tilde{\gamma}(u) \ni f, \quad -k_1 u - k_2 \partial u / \partial \nu \in \tilde{\beta}(u \mid_\Gamma)
\]

\textit{has a solution} \( u \in H^2(G) \).

**Proof.** The boundary condition can be written as

\[-\partial u / \partial \nu \in k_1 k_2^{-1} u + k_2^{-1} \beta(u \mid_\Gamma).
\]

On the other hand, it can be verified that \( k_1 k_2^{-1} l + k_2^{-1} \beta \) is a maximal
monotone mapping in $R$, using the well-known fact that a monotone mapping $U$ in a Hilbert space $H$ is maximal if and only if for all $\lambda > 0$ the range of $I + \lambda U$ is the whole of $H$ (see e.g. [2]).

REFERENCES


