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CONCERNING THE HAHN-BANACH THEOREM[1]

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ABSTRACT. We establish an intrinsic characterization for those normed spaces having the extension property that applies equally to spaces with real, complex, or quaternionic scalars. Nachbin's characterization for real spaces via the binary intersection property follows as a special case. The method also yields a proof of the Hahn-Banach theorem that does not depend on the choice of scalar field.

The original motivation for this work was dissatisfaction with the traditional proof of the Hahn-Banach theorem for complex normed linear spaces. Several years ago we found an alternate argument that applies equally to real, complex, or quaternionic scalars, and noted that the technique of our proof suggests an intrinsic characterization of those normed spaces with the extension property (see Theorems 3 and 5). The referee of the present write-up has pointed out that similar results appear in the preprint [5] by O. Hustad.

We recall that the Hahn-Banach extension theorem for *real* spaces dates from papers by H. Hahn [3] in 1927 and S. Banach [1] in 1929, while the familiar trick deriving the theorem for other scalars by reduction to the real case was not forthcoming until 1938: H. F. Bohnenblust and A. Sobczyk [2] (complex scalars); G. A. Soukhomlinoff [9] (complex or quaternionic scalars). The number of intervening years indicates that this trick, simple though it may be, is neither natural nor inevitable. In fact, we may base the extension theorem on the following lemma, establishing a simple property shared by the three scalar fields used with normed linear spaces: the reals $\mathbf{R}$, the complex numbers $\mathbf{C}$, and the quaternions $\mathbf{Q}$. In preparation we merely note that each of these scalar fields F has a conjugation (indicated as usual by a bar) such that $z\bar{z} = |z|^2$, and that the distance $|z - w|$ in F is Euclidean.

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Lemma 1. *Each scalar field F has the following "intersection property": if $c_k \in F$ and $r_k > 0$, the balls*

$$B_{r_k}(c_k) = \{z \in F\colon |z - c_k| \le r_k\} \qquad (k = 1, 2, \cdots, n)$$

have nonempty intersection provided $|\Sigma_1^n z_k c_k| \le \Sigma_1^n |z_k| r_k$ for any $z_k \in F$ such that $\Sigma_1^n z_k = 0$.

Proof. Let m be the minimum value of the function f defined on F by

$$f(z) = \max_{1 \le k \le n} \frac{|z - c_k|}{r_k},$$

and let $\beta \in F$ be a point at which this minimum is attained. We may assume that $|\beta - c_k|/r_k = m$ for $1 \le k \le p$, while $|\beta - c_k|/r_k < m$ for $p < k \le n$. Now β must lie in the convex hull K of $c_1, c_2, \ldots, c_p$. Otherwise, if β_0 is the point in K closest to β, then the dot product $(\beta - \beta_0) \cdot (c_k - \beta_0)$ is non-positive for $1 \le k \le p$. Hence, for any z between β and β_0 and $1 \le k \le p$, $|z - c_k| < |\beta - c_k|$ so that $|z - c_k|/r_k < m$. Since this last inequality holds also for $p < k \le n$ when z is close to β, we would have $f(z) < m$, a contradiction.

We have, then, $\beta = \Sigma_1^P \lambda_k c_k$, where $\lambda_k \ge 0$ and $\Sigma_1^P \lambda_k = 1$. Set $z_k = \lambda_k \overline{(\beta - c_k)}$ and note that $\Sigma_1^P z_k = 0$, and that $z_k(\beta - c_k) = \lambda_k |\beta - c_k|^2 \ge 0$. Hence,

$$m \sum_1^P |z_k| r_k = \sum_1^P |z_k|\,|\beta - c_k| = \left|\sum_1^P z_k(\beta - c_k)\right| = \left|\sum_1^P z_k c_k\right| \le \sum_1^P |z_k| r_k,$$

so that $m \le 1$. Q.E.D.

Theorem 2. *If S is a subspace of the normed linear space $(X, \|\cdot\|)$ with scalars F, and $\phi\colon S \to F$ is a (linear) functional such that $(\forall s \in S)$ $|\phi(s)| \le \|s\|$, then there exists a functional $\tilde{\phi}\colon X \to F$ extending ϕ and such that $(\forall x \in X)$ $|\tilde{\phi}(x)| \le \|x\|$.*

Proof. By the usual induction (possibly transfinite, if X/S is big), we reduce to the "one-step extension problem" where X is spanned by S and a single additional vector u. We must simply find a value $\alpha \in F$ for $\tilde{\phi}(u)$, and the condition on α is easily seen to be the following: $(\forall s \in S)$ $|\alpha - \phi(s)| \le \|u - s\|$. In the notation of Lemma 1, we must show that the balls $\{B_{\|u-s\|}(\phi(s))\colon s \in S\}$ have nonempty intersection. Since each ball is compact we need only show that any finite collection of the balls has nonempty intersection. By Lemma 1, it suffices to note that, whenever $s_k \in S$, $z_k \in F$, and $\Sigma_1^n z_k = 0$,

$$\left|\sum_1^n z_k \phi(s_k)\right| = \left|\phi\left(\sum_1^n z_k s_k\right)\right| \leq \left\|\sum_1^n z_k s_k\right\| = \left\|\sum_1^n z_k(s_k - u)\right\| \leq \sum_1^n |z_k| \, \|u - s_k\|.$$

Q.E.D.

In the study of extension theorems for linear transformations more general than functionals, the following notion has received considerable attention: a normed linear space $(Y, \|\cdot\|)$ (with scalar field F) is said to have the *extension property* (EP) provided that, for any normed linear space $(X, \|\cdot\|)$ (with scalars F), subspace S of X, and linear transformation T: $S \to Y$ such that $(\forall s \in S)\ \|T(s)\| \leq \|s\|$, there exists a linear extension $\tilde{T}$: $X \to Y$ such that $(\forall x \in X)\ \|\tilde{T}(x)\| \leq \|x\|$. For example, Theorem 2 says that any one-dimensional Y (which is thus isomorphic to F) has the (EP).

Several characterizations of those spaces Y having the (EP) are available. It is known, for example, that Y has the (EP) if, and only if, Y has the *projection property*: there is a contractive projection onto Y from any normed linear space Z containing Y as a subspace. In another direction, J. L. Kelley [6] and M. Hasumi [4] have shown that the spaces Y with the (EP) are precisely those isomorphic to $C(H)$, the space of continuous scalar functions on some Stonian space H. These characterizations are not, however, intrinsic; that is, they are not expressed simply in terms of the geometry of the space Y.

L. Nachbin [8] gives us an elegant intrinsic description of those *real* normed spaces Y having the (EP). Here, and later, $\mathcal{B}$ always denotes a collection of balls in Y; that is, each $B \in \mathcal{B}$ has the form

$$B = B_r(c) = \{y \in Y: \ \|y - c\| \leq r\},$$

for some $c \in Y$ and $r \geq 0$. Nachbin shows that, for $(Y, \|\cdot\|)$ with scalars $F = \mathbf{R}$, Y has the (EP) if, and only if, Y has the *binary intersection property*:

$$\text{(2-IP)} \qquad (\forall B_1, B_2 \in \mathcal{B}) B_1 \cap B_2 \neq \emptyset \Rightarrow \bigcap \mathcal{B} \neq \emptyset.$$

For further work on the case $F = \mathbf{R}$, the reader might consult the memoir [7] by J. Lindenstrauss. The analysis of the Hahn-Banach theorem that we have presented in Lemma 1 and Theorem 2 suggests an intrinsic characterization of the (EP) which is valid for *any* of the scalar fields $F = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{Q}$. Theorems 3 and 5, below, state precise results of this nature. In Corollary 6 we will see that Nachbin's result follows naturally upon adding the assumption that $F = \mathbf{R}$.

We shall say that a normed linear space Y, with scalars F, has the *intersection property* (IP) provided it mimics the behavior of the scalar fields F established in Lemma 1. More precisely, we say Y has (IP) if any collection $\mathcal{B}$ of balls in Y has nonempty intersection ($\bigcap \mathcal{B} \neq \emptyset$) whenever

$$(1) \qquad B_{r_k}(c_k) \in \mathcal{B},\ z_k \in F, \text{ and } \sum_1^n z_k = 0 \Longrightarrow \left\| \sum_1^n z_k c_k \right\| \leq \sum_1^n |z_k| r_k.$$

We remark that it is a simple matter to show that $(\bigcap \mathcal{B} \neq \emptyset) \Longrightarrow (1)$, for any normed space Y.

Theorem 3. *Let* $(Y, \|\cdot\|)$ *be a normed linear space with scalars* F. *Then* Y *has the extension property* (*EP*) *if, and only if, it has the intersection property* (*IP*).

Proof. To establish the (EP) for $(Y, \|\cdot\|)$ we consider a linear transformation $T\colon S \to Y$ such that $(\forall s \in S)\ \|T(s)\| \leq \|s\|$, and, following the model of the proof of Theorem 2, we must simply show that $\bigcap \mathcal{B} \neq \emptyset$ where

$$\mathcal{B} = \{B_{\|u-s\|}(T(s))\colon s \in S\}.$$

Since we are assuming (IP), we need only verify (1) for this $\mathcal{B}$; but if $\Sigma z_k = 0$ and $s_k \in S$,

$$\left\| \sum_k z_k T(s_k) \right\| = \left\| T\left(\sum_k z_k s_k\right) \right\| \leq \left\| \sum_k z_k s_k \right\|$$
$$= \left\| \sum_k z_k (s_k - u) \right\| \leq \sum_k |z_k|\, \|u - s_k\|.$$

On the other hand, suppose Y has the (EP) and that $\mathcal{B}$ satisfies (1). Consider the vector space X over F defined by $X = \{\Sigma_k z_k c_k\colon z_k \in F,\ c_k \in \mathcal{C}\}$, where $\mathcal{C} = \{c \in Y\colon B_r(c) \in \mathcal{B} \text{ for some } r \geq 0\}$. Introduce the seminorm $\|\cdot\|_0$ on X as follows:

$$\|x\|_0 = \inf \left\{ \sum_k |z_k| r_k\colon x = \sum_k z_k c_k \text{ and } B_{r_k}(c_k) \in \mathcal{B} \right\},$$

and identify elements of X at distance 0 with respect to the seminorm. The set $S = \{\Sigma_k z_k c_k\colon z_k \in F,\ \Sigma_k z_k = 0,\ c_k \in \mathcal{C}\}$ is a subspace of X and (1) simply assures that $(\forall s \in S)\ \|s\| \leq \|s\|_0$. Hence the identity map T on S is contractive, and by the (EP) T has a contractive linear extension $\tilde{T}$ to X. Note that, for any pair $c, c' \in \mathcal{C}$, $c - c' \in S$, so that

$$\tilde{T}(c) - \tilde{T}(c') = \tilde{T}(c - c') = T(c - c') = c - c'.$$

Let y denote the element of Y such that $(\forall c \in \mathcal{C})\ y = c - \tilde{T}(c)$. It follows that

$$B_r(c) \in \mathcal{B} \Longrightarrow \|y - c\| = \|\tilde{T}(c)\| \leq \|c\|_0 \leq 1 \cdot r,$$

so that $y \in \bigcap \mathcal{B}$. Q.E.D.

Lemma 4. *In any normed linear space Y with scalars F,* (1) *is equivalent to the following statement concerning $\mathcal{B}$:*

(2) $z_k \in F,\ |z_k| = 1,\ \sum_k z_k = 0,$ *and* $B_{r_k}(c_k) \in \mathcal{B} \Longrightarrow \left\|\sum_k z_k c_k\right\| \leq \sum_k r_k.$

Proof. Certainly (1) $\Longrightarrow$(2). On the other hand, (1) follows by continuity once it is known for z_k which are "commensurable" in the sense that, for some $\epsilon > 0$ and integers n_k, $|z_k| = n_k\epsilon$. In this case, let $w_k = z_k/n_k\epsilon$, and observe that $|w_k| = 1$, $\Sigma_k n_k w_k = (1/\epsilon)\Sigma_k z_k = 0$. Thus (2) ensures that $\|\Sigma_k n_k w_k c_k\| \leq \Sigma_k n_k r_k$, so that

$$\left\|\sum_k z_k c_k\right\| = \epsilon\left\|\sum_k n_k w_k c_k\right\| \leq \sum_k \epsilon n_k r_k = \sum_k |z_k| r_k.$$

Lemma 4 allows us to replace Theorem 3 by the following variant, from which Nachbin's theorem is conveniently derived as a special case.

Theorem 5. *Let $(Y, \|\cdot\|)$ be a normed linear space with scalars F. Then Y has the (EP) if, and only if, it has the intersection property*

(IP_1) $\qquad \bigcap \mathcal{B} \neq \varnothing$ *whenever* $\mathcal{B}$ *satisfies* (2).

Corollary 6 (Nachbin). *A real normed linear space has the (EP) if, and only if, it has the binary intersection property* (2-IP).

Proof. We simply check that, when $F = \mathbf{R}$, (2) is equivalent to

(3) $$(\forall B_1, B_2 \in \mathcal{B})\ B_1 \cap B_2 \neq \varnothing.$$

Setting $z_1 = 1$, $z_2 = -1$ in (2) we obtain $\|c_1 - c_2\| \leq r_1 + r_2$, which simply means that $B_{r_1}(c_1) \cap B_{r_2}(c_2) \neq \varnothing$. Thus (for any F) (2) $\Longrightarrow$(3), and hence (2-IP) $\Longrightarrow$ (EP). On the other hand, if $\Sigma z_k = 0$, $|z_k| = 1$, and $z_k \in \mathbf{R}$, we may clearly assume that $z_1 = 1$, $z_2 = -1$, $z_3 = 1, \ldots, z_{2p} = -1$. Assuming (3), we have $\|c_{2j-1} - c_{2j}\| \leq r_{2j-1} + r_{2j}$, and hence

$$\left\|\sum z_k c_k\right\| = \left\|\sum_j (c_{2j-1} - c_{2j})\right\|$$

$$\leq \sum_j \|c_{2j-1} - c_{2j}\| \leq \sum_j (r_{2j-1} + r_{2j}) = \sum_k r_k. \quad \text{Q.E.D.}$$

Remark. While $\mathbf{R}$ has the binary intersection property (2-IP), $\mathbf{C}$ has the *ternary* intersection property (3-IP): $\bigcap \mathfrak{B} \neq \emptyset$ provided that

$$(\forall B_1, B_2, B_3 \in \mathfrak{B})\ B_1 \cap B_2 \cap B_3 \neq \emptyset. \tag{4}$$

This is a special case of Helly's theorem on convex sets in $\mathbf{R}^n$. In view of Nachbin's theorem, it is natural to ask whether the *complex* spaces with the (EP) can be characterized by (3-IP). It is not hard to show that (EP) $\Rightarrow$ (3-IP) by imbedding Y as a subspace of the bounded $\mathbf{C}$-valued functions on U^* = the unit ball in the dual space Y^*. We do not know whether (3-IP) $\Rightarrow$ (EP) when $F = \mathbf{C}$, but in any case we cannot proceed as in our proof of Corollary 6, since examples show that (4) need not follow from (2).

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