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AN ASYMPTOTIC FUGLEDE THEOREM

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ABSTRACT. The main result is that if an operator B on Hilbert space "almost" commutes with a normal operator N, then B almost commutes with N^* as well. The Theorem is then extended to a Putnam-like version which states that if B almost intertwines two normal operators, then it almost intertwines continuous functions of those operators.

Let N be a normal operator on Hilbert space with spectral measure E, and let B be any bounded operator. Two common equivalent formulations of the Fuglede theorem (for example, see [3]) are the following:

(i) If $NB = BN$, then $N^*B = BN^*$.

(ii) If $NB = BN$, then $E(S)B = BE(S)$ for all Borel sets S.

Is an approximate version of these statements true? That is, if $\|NB - BN\|$ is "small," is $\|N^*B - BN^*\|$ small? Is $\|E(S)B - BE(S)\|$ small? Bastian and Harrison [1] showed that the answer to the second question is "no," by constructing a normal operator N, a Borel set S, and a sequence $\{B_n\}$ for which $\|B_n\| = 1$ for all n and $\|NB_n - B_nN\| \to 0$, but $\|E(S)B_n - B_nE(S)\| = 1$ for all n. Johnson and Williams [4] have shown the existence of a normal operator N and a sequence $\{B_n\}$ such that $\|B_nN - NB_n\| \to 0$ but $\|N^*B_n - B_nN^*\| \geq 1$ for all n. Hence the answer to the first question is also negative.

In the Johnson-Williams example, the operators $\{B_n\}$ are not uniformly bounded. The purpose of this note is to show that in case $\|B\|$ is under control, then the answer to the first question is affirmative. Our techniques were inspired by those of Rosenblum [5].

Theorem. *Let N be a normal operator. If K and ϵ are greater than* 0, *then there is δ greater than* 0 *such that $\|B\| \leq K$ and $\|NB - BN\| \leq \delta$ imply $\|N^*B - BN^*\| \leq \epsilon$.*

Proof. Without loss of generality we assume that $\|N\| \leq 1$, and that $K = 1$. Let B be an operator with $\|B\| \leq 1$, and set $\|NB - BN\| = \eta$. A

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well-known inductive argument shows that

$$\|N^k B - BN^k\| \le k\|N\|^{k-1}\|NB - BN\| \le k\eta,$$

for positive integers k. Thus, for any complex number λ,

$$\|e^{i\bar{\lambda}N}B - Be^{i\bar{\lambda}N}\| = \left\|\sum_{k=0}^{\infty} \frac{(i\bar{\lambda})^k}{k!}(N^k B - BN^k)\right\|$$

$$\le \sum_{k=0}^{\infty} \frac{|\lambda|^k}{k!}\|N^k B - BN^k\| \le \sum_{k=0}^{\infty} \frac{|\lambda|^k}{k!} k\eta$$

$$= \eta|\lambda| \sum_{k=1}^{\infty} \frac{|\lambda|^{k-1}}{(k-1)!} = \eta|\lambda|e^{|\lambda|}.$$

Similarly we can show that $e^{|\lambda|}$ bounds each of the norms $\|e^{i\bar{\lambda}N}\|$, $\|e^{-i\lambda N^*}\|$, and $\|e^{i\lambda N^*}\|$. Hence

$$\|e^{i\lambda N^*}(e^{i\bar{\lambda}N}B - Be^{i\bar{\lambda}N})e^{-i\bar{\lambda}N}e^{-i\lambda N^*}\| \le \eta|\lambda|e^{4|\lambda|}.$$

Since N is normal we can add exponents to get

$$\|e^{i(\bar{\lambda}N + \lambda N^*)}Be^{-i(\bar{\lambda}N + \lambda N^*)} - e^{i\lambda N^*}Be^{-i\lambda N^*}\| \le \eta|\lambda|e^{4|\lambda|}$$

or

$$\|e^{i\lambda N^*}Be^{-i\lambda N^*}\| \le \|e^{i(\bar{\lambda}N + \lambda N^*)}Be^{-i(\bar{\lambda}N + \lambda N^*)}\| + \eta|\lambda|e^{4|\lambda|}.$$

But $\bar{\lambda}N + \lambda N^*$ is Hermitian, so $e^{i(\bar{\lambda}N + \lambda N^*)}$ is unitary, and the first term on the right-hand side of the last inequality is equal to $\|B\|$ for all values of λ. Thus

$$\|e^{i\lambda N^*}Be^{-i\lambda N^*}\| \le \|B\| + \eta|\lambda|e^{4|\lambda|} \le 1 + \eta|\lambda|e^{4|\lambda|}.$$

The function ϕ defined by $\phi(\lambda) = e^{i\lambda N^*}Be^{-i\lambda N^*}$ is an analytic operator-valued function of λ. By Cauchy's integral formula we have

$$\phi'(0) = \frac{1}{2\pi i}\int_\Gamma \frac{\phi(\lambda)}{\lambda^2}\,d\lambda$$

where Γ is any closed Jordan curve whose interior contains the origin. In case Γ is a circle of radius r with center at the origin we get the estimate

$$\|\phi'(0)\| \le \frac{2\pi r}{2\pi}\cdot\frac{1}{r^2}\cdot\max_\Gamma\|\phi(\lambda)\| \le \frac{1}{r}(1 + \eta r e^{4r}) = \frac{1}{r} + \eta e^{4r}.$$

Notice that

$$\phi'(\lambda) = iN^* e^{i\lambda N^*} Be^{-i\lambda N^*} - ie^{i\lambda N^*} BN^* e^{-i\lambda N^*},$$

so that $\|\phi'(0)\| = \|N^*B - BN^*\|$. Hence

$$\|N^*B - BN^*\| \leq 1/r + \eta e^{4r},$$

and this equation holds for all $r > 0$. Thus if $\epsilon > 0$, choose $r = 2/\epsilon$ and choose $\delta = \frac{1}{2}\epsilon e^{-4r}$. Then when $\|NB - BN\| \leq \delta$, that is, $\eta \leq \delta$, we have

$$\|N^*B - BN^*\| \leq \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon e^{-4r} e^{4r} = \epsilon.$$

This completes the proof.

S. K. Berberian's (by now well-known) trick [2] allows us to extend the result to a Putnam-like version:

Corollary 1. *Let N_1 and N_2 be normal. If K and ϵ are greater than zero, there exists δ greater than zero such that $\|B\| \leq K$ and $\|N_1B - BN_2\| \leq \delta$ imply $\|N_1^*B - BN_2^*\| \leq \epsilon$.*

Proof. Let $\mathcal{H}$ be the Hilbert space on which N_1, N_2, and B act. Define operators $\tilde{N}$ and $\tilde{B}$ on the space $\mathcal{H} \oplus \mathcal{H}$ by

$$\tilde{N} = \begin{pmatrix} N_2 & 0 \\ 0 & N_1 \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}.$$

Then $\|\tilde{B}\| = \|B\|$, $\|N_1B - BN_2\| = \|\tilde{N}\tilde{B} - \tilde{B}\tilde{N}\|$, and $\|N_1^*B - BN_2^*\| = \|\tilde{N}^*\tilde{B} - \tilde{B}\tilde{N}^*\|$. Application of the Theorem to $\tilde{N}$ and $\tilde{B}$ gives the corollary.

Corollary 2. *Let N_1 and N_2 be normal and let ψ be a complex-valued function continuous on the union of the spectra of N_1 and N_2. If K and ϵ are greater than zero, then there is δ greater than zero such that $\|B\| \leq K$ and $\|N_1B - BN_2\| \leq \delta$ imply that $\|\psi(N_1)B - B\psi(N_2)\| \leq \epsilon$.*

Proof. The technique used to prove Corollary 1, plus the observation that $\psi(N_1 \oplus N_2) = \psi(N_1) \oplus \psi(N_2)$, shows that we may assume that $N_1 = N_2 = N$; without loss of generality we assume as before that $\|N\| = 1$. We use a three-step proof.

Step one. In case $\psi(N)$ is a monomial in N and N^*, say $\psi(N) = N^k N^{*j}$, we have

$$\|N^kN^{*j}B - BN^kN^{*j}\| \leq \|N^kN^{*j}B - N^kBN^{*j}\| + \|N^kBN^{*j} - BN^kN^{*j}\|$$
$$\leq \|N^{*j}B - BN^{*j}\| + \|N^kB - BN^k\|,$$

since N has norm 1. By the relation used at the beginning of the proof of the Theorem,

$$\|N^kN^{*j}B - BN^kN^{*j}\| \leq j\|N^*\|^{j-1}\|N^*B - BN^*\| + k\|N\|^{k-1}\|NB - BN\|$$
$$= j\|N^*B - BN^*\| + k\|NB - BN\|.$$

Choose δ' so that $\|B\| \leq K$ and $\|NB - BN\| \leq \delta'$ imply $\|N^*B - BN^*\| \leq \epsilon/2j$, and let $\delta = \min(\delta', \epsilon/2k)$. Then if $\|B\| \leq K$ and $\|NB - BN\| \leq \delta$ we have

$$\|N^kN^{*j}B - BN^kN^{*j}\| \leq j \cdot \epsilon/2j + k \cdot \epsilon/2k = \epsilon.$$

Step two. In case $\psi(N)$ is a polynomial in N and N^*, say $\psi(N) = \sum_{k+j\leq m}\alpha_{kj}N^kN^{*j}$, we have

$$\|\psi(N)B - B\psi(N)\| \leq \sum_{k+j\leq m} |\alpha_{kj}|\|N^kN^{*j}B - BN^kN^{*j}\|.$$

By Step one, choose δ_{kj} greater than zero so that $\|B\| \leq K$ and $\|NB - BN\| \leq \delta_{kj}$ imply $\|N^kN^{*j}B - BN^kN^{*j}\| \leq \epsilon/|\alpha_{kj}|m(m+1)$. Let $\delta = \min\{\delta_{kj}: k + j \leq m\}$. Then $\delta > 0$ and the conditions $\|B\| \leq K$ and $\|NB - BN\| \leq \delta$ imply

$$\|\psi(N)B - B\psi(N)\| \leq \sum_{k+j\leq m} |\alpha_{kj}| \cdot \frac{\epsilon}{|\alpha_{kj}|m(m+1)}$$
$$\leq \sum_{k+j\leq m} \frac{\epsilon}{m(m+1)} \leq \epsilon.$$

Step three. Now let ψ be any function continuous on the spectrum of N. Then there is a polynomial $p(\lambda, \bar{\lambda})$ such that $\sup_{\lambda\in\Lambda(N)}|p(\lambda, \bar{\lambda}) - \psi(\lambda)| \leq \epsilon/4K$, where $\Lambda(N)$ is the spectrum of N. We then have

$$\|\psi(N)B - B\psi(N)\| \leq \|\psi(N)B - p(N, N^*)B\| + \|p(N, N^*)B - Bp(N, N^*)\|$$
$$+ \|Bp(N, N^*) - B\psi(N)\|$$
$$\leq 2\|B\| \cdot \|\psi(N) - p(N, N^*)\| + \|p(N, N^*)B - Bp(N, N^*)\|.$$

Finally, by Step two we can choose δ greater than zero so that $\|B\| \leq K$ and $\|NB - BN\| \leq \delta$ imply that $\|p(N, N^*)B - Bp(N, N^*)\| \leq \epsilon/2$. The condition defining the polynomial p, plus the spectral theorem, imply that

$\|\psi(N) - p(N, N^*)\| \leq \epsilon/4K$. Hence for $\|B\| \leq K$ and $\|NB - BN\| \leq \delta$ we have

$$\|\psi(N)B - B\psi(N)\| \leq 2K \cdot \epsilon/4K + \epsilon/2 = \epsilon.$$

The proof is complete.

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