

## AN ELEMENTARY INVARIANT DEFINITION OF THE FUNCTIONS OF BIDEGREE $(p, q)$

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ABSTRACT. The alternating  $r$ -linear complex-valued functions of bidegree  $(p, q)$ ,  $p + q = r$ , are usually defined on a complex vector space  $V$  as the span of the elements  $g_{i_1} \wedge \cdots \wedge g_{i_p} \wedge \bar{g}_{j_1} \wedge \cdots \wedge \bar{g}_{j_q}$ , where  $\{g_i : i \in I\}$  is a basis for  $V^*$ , or by means of a representation of the exterior power of a direct sum. The former definition is not a priori invariant under coordinate changes and not easily adaptable to analysis on infinite-dimensional spaces, and the latter one rests on a rather involved abstract construction. Here it is shown how to give a new coordinate-free definition of the  $(p, q)$  functions by means of a simple identity which characterizes them by their action as  $r$ -linear maps on  $V$ . It seems well adapted for analysis on infinite-dimensional spaces.

The space  $\wedge^r V'$  of alternating real-multilinear maps  $\omega : V \times \cdots \times V$  ( $r$  factors)  $\rightarrow \mathbb{C}$  of degree  $r$  defined on a complex vector space  $V$  is a standard object in complex analysis. It is very important that this space has a canonical decomposition

$$(1) \quad \wedge^r V' = \sum_{p+q=r} \wedge^{p,q} V'$$

as a direct sum of its subspaces  $\wedge^{p,q} V'$  of maps of bidegree  $(p, q)$  [6, p. 12] because (1) induces a corresponding decomposition of the space of complex-valued differential forms of degree  $r$  which yields the relation  $d = \partial + \bar{\partial}$  between the ordinary exterior differentiation  $d$  and the Cauchy-Riemann operators  $\partial$  and  $\bar{\partial}$  of complex analysis. The subspaces  $\wedge^{p,q} V'$  are often defined in terms of a basis  $\{g_i : i \in I\}$  for  $V^*$  (the usual dual space of complex-linear maps) as the span

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$$(2) \quad \wedge^{p,q} V' = \text{span} \{ g_{i_1} \wedge \cdots \wedge g_{i_p} \wedge \bar{g}_{j_1} \wedge \cdots \wedge \bar{g}_{j_q} : i_k, j_l \in I \},$$

a definition which has some unsatisfactory aspects. Two of them are basically esthetic: Its independence of a choice of basis requires some proof, and it does not define the elements of  $\wedge^{p,q} V'$  in terms of their behavior as maps. A more practical difficulty is that its adaptation to a decomposition of the space of continuous alternating functions on a topological vector space is cumbersome and perhaps in some cases even inappropriate (see Remark 1 below). A common alternative is to note that the space  $\wedge^1 V'$  of complex-valued real-linear maps on  $V$  is the direct sum

$$(3) \quad \wedge^1 V' = V^* \oplus \overline{V^*}$$

(in which  $\overline{V^*}$  is the subspace of all conjugate-linear maps) and define  $\wedge^{p,q} V'$  by interpreting a construction of abstract exterior algebra [2, p. 121] which represents the  $r$ th exterior power of a direct sum (see Remark 2 below). This procedure is rather involved, and departs from a concrete viewpoint. In this note a simple coordinate-free definition of the space  $\wedge^{p,q} V'$  is given by means of the identity (4) which characterizes the functions of bidegree  $(p, q)$  by their action as multilinear functions on  $V$ . (Remark 3 below compares this with another such characterization which is based on a different representation of  $\wedge^r V'$  as a space of alternating maps.)

For each  $v = (v_1, \dots, v_r) \in V \times \cdots \times V$  and each  $j \leq r$  let  $v^j = (v_1, \dots, v_{j-1}, iv_j, v_{j+1}, \dots, v_r)$ . Thus  $v \rightarrow v^j$  leaves fixed all but the  $j$ th coordinate of  $v$ , which it replaces by  $iv_j$  ( $i^2 = -1$ ).

**Definition 1.** A function  $\omega \in \wedge^r V'$  is of *bidegree*  $(p, q)$  if  $p + q = r$  and for all  $v \in V \times \cdots \times V$ ,

$$(4) \quad i(p - q)\omega(v) = \sum_{j=1}^r \omega(v^j).$$

The set of all such elements of  $\wedge^r V'$  is denoted  $\wedge^{p,q} V'$ . It is clearly a complex subspace of  $\wedge^r V'$ . Moreover, the definition is "functorial" in the sense that the pull-back map  $T^*: \wedge^r W' \rightarrow \wedge^r V'$  induced from a complex-linear map  $T: V \rightarrow W$  by  $(T^*\omega)(v) = \omega(Tv_1, \dots, Tv_r)$  preserves the  $(p, q)$  functions:  $T^*(\wedge^{p,q} W') \subset \wedge^{p,q} V'$ .

H. B. Lawson and J. Simons have used this definition in the special case  $p = q$  [3, p. 445]. In this case one has nontrivial real valued  $(p, p)$  functions. Remark 4 compares Definition 1 with a similar but less direct description of  $\wedge^{p,q} V'$ .

Exposition of the basic properties of this definition will be organized around three different proofs of the following central result.

**Theorem 1.** *The direct sum decomposition (1) holds, with  $\bigwedge^{p,q}V'$  defined by (4).*

The first proof is designed to motivate the definition of  $\bigwedge^{p,q}V'$  by showing how (1) follows in a fairly straightforward way from an analogous decomposition (5) of the space  $\bigotimes^r V'$  of all complex-valued multilinear (over  $\mathbf{R}$ ) maps on  $V$ . The second proof makes use of a basis for  $V^*$  and recovers as a by-product the equivalence of Definition 1 and (2). The third proof is direct and based only on Definition 1.

To begin the first proof, it is clear from (4) that the subspaces  $\bigwedge^{p,q}V'$  are independent, for since  $p + q = r$  the bidegree  $(p, q)$  is determined by the difference  $p - q$ , and the right side of (4) is independent of  $p$  or  $q$ . To show that these subspaces span  $\bigwedge^r V'$  it is claimed that the space  $\bigotimes^r V'$  defined above is a direct sum

$$(5) \quad \bigotimes^r V' = \sum_E \bigotimes_E^r V'$$

of its subspaces  $\bigotimes_E^r V'$  defined for each subset  $E$  of  $\{1, \dots, r\}$  as the set of all maps  $\omega \in \bigotimes^r V'$  which are complex-linear in each coordinate indexed by a number in  $E$  and conjugate-linear in each coordinate indexed by a number not in  $E$ . Thus an element of  $\bigotimes_{\{1\}}^2 V'$  is a map  $v \rightarrow \omega(v_1, v_2)$  which is complex-linear in  $v_1$  and conjugate-linear in  $v_2$ . If  $\omega \in \bigotimes^r V'$  its component in  $\bigotimes_E^r V'$  is denoted  $\omega_E$ .

A complete proof of (5) will not be given. However, note that (3) is the special case of it for  $r = 1$ , because it is clear that  $V^* = \bigotimes_{\{1\}}^1 V'$  and  $\overline{V^*} = \bigotimes_{\emptyset}^1 V'$ . A direct proof of it for this special case can be given [6, p. 12] by solving for the components  $\omega_{\{1\}}$  and  $\omega_{\emptyset}$  of  $\omega$  in  $\omega(v) = \omega_{\{1\}}(v) + \omega_{\emptyset}(v)$  and  $i\omega(iv) = -\omega_{\{1\}}(v) + \omega_{\emptyset}(v)$ , obtaining the formulas  $\omega_{\{1\}}(v) = \frac{1}{2}(\omega(v) - i\omega(iv))$  and  $\omega_{\emptyset}(v) = \frac{1}{2}(\omega(v) + i\omega(iv))$ . It is then shown that these formulas define elements of  $V^*$  and  $\overline{V^*}$ , respectively. This method of proof can be generalized to  $\bigotimes^r V'$  to yield for the  $2^r$  values  $\omega_E(v)$  a system of  $2^r$  linear equations whose matrix can be shown to have orthogonal nonzero rows. Hence it can be explicitly inverted to give solutions  $\omega_E \in \bigotimes^r V'$  such that  $\omega = \sum_E \omega_E$ . By means of the formulas obtained for the  $\omega_E$  it is then shown that each  $\omega_E \in \bigotimes_E^r V'$ .

One is led to (4) and a proof of (1) by seeking the significance of (5) for an element  $\omega \in \wedge^r V' \subset \otimes^r V'$ . Of course its components  $\omega_E$  need not be alternating; for example, the only alternating element of  $\otimes_{\{1\}}^2 V'$  is zero. However, grouping the terms of (5) as  $\omega_p = \sum_{|E|=p} \omega_E$  (where  $|E|$  denotes the cardinality of  $E$ ) gives  $\omega = \sum_{p=0}^r \omega_p$ , and each  $\omega_p$  is alternating: For if  $\sigma$  is a permutation of  $\{1, \dots, r\}$  then

$$\sum_{p=0}^r \sum_{|E|=p} \text{sgn } \sigma \omega_E(v) = \text{sgn } \sigma \omega(v) = \omega(\sigma v) = \sum_{p=0}^r \sum_{|E|=p} \omega_E(\sigma v)$$

where  $\sigma v = (v_{\sigma(1)}, \dots, v_{\sigma(r)})$ .

Now each map  $v \rightarrow \omega_E(\sigma v)$  is an element of  $\otimes_{\sigma^{-1}(E)}^r V'$  and  $E \rightarrow \sigma^{-1}(E)$  is a bijection of the class of sets  $E$  with  $p$  elements. Thus by uniqueness in (5)  $\text{sgn } \sigma \omega_{\sigma^{-1}(E)}(v) = \omega_E(\sigma v)$  for all  $v$ , and by summing this over all sets  $E$  of cardinality  $p$  one gets

$$\text{sgn } \sigma \omega_E \sum_{|E|=p} \omega_E(v) = \sum_{|E|=p} \omega_E(\sigma v).$$

This shows that  $\omega_p \in \wedge^r V'$ . Finally, each  $\omega_p$  satisfies the identity (4):

$$\begin{aligned} \sum_{j=1}^r \omega_p(v^j) &= \sum_{|E|=p} \sum_{j=1}^r \omega_E(v^j) = \sum_{|E|=p} \left( \sum_{j \in E} \omega_E(v^j) + \sum_{j \notin E} \omega_E(v^j) \right) \\ &= \sum_{|E|=p} \left( i \sum_{j \in E} \omega_E(v) - i \sum_{j \notin E} \omega_E(v) \right) \\ &= i(p - q) \sum_{|E|=p} \omega_E(v) = i(p - q)\omega_p(v). \end{aligned}$$

The first proof of Theorem 1 is complete.

The second proof of it depends on another basic property.

**Theorem 2.** *If  $\omega \in \wedge^{p,q} V'$  and  $\eta \in \wedge^{s,t} V'$  then  $\omega \wedge \eta \in \wedge^{p+q,s+t} V'$ .*

Here  $\omega \wedge \eta$  is the usual wedge product defined in this instance [5, p. 7-4] by

$$\begin{aligned} \omega \wedge \eta(v) &= (p + q + s + t)! / (p + q)!(s + t)! \text{Alt } \omega \otimes \eta(v) \\ &= 1 / (p + q)!(s + t)! \sum_{\sigma \in S_r} \text{sgn } \sigma (\omega \otimes \eta)(\sigma v), \end{aligned}$$

where  $S_r$  is the group of all permutations of  $\{1, \dots, r\}$ .

To prove Theorem 2 let  $C = 1 / (p + q)!(s + t)!$ ,  $r = p + q + s + t$ . Then

$$\begin{aligned}
 \sum_{j=1}^u (\omega \wedge \eta)(v^j) &= C \sum_{j=1}^u \sum_{\sigma \in S_u} \text{sgn } \sigma \omega(v_{\sigma(1)}^j, \dots, v_{\sigma(r)}^j) \eta(v_{\sigma(r+1)}^j, \dots, v_{\sigma(u)}^j) \\
 &= C \sum_{\sigma \in S_u} \text{sgn } \sigma \left[ \left( \sum_{j \in \{\sigma(1), \dots, \sigma(r)\}} \omega(v_{\sigma(1)}^j, \dots, v_{\sigma(r)}^j) \right) \eta(v_{\sigma(r+1)}, \dots, v_{\sigma(u)}) \right. \\
 &\quad \left. + \omega(v_{\sigma(1)}, \dots, v_{\sigma(r)}) \sum_{j \in \{\sigma(r+1), \dots, \sigma(u)\}} \eta(v_{\sigma(r+1)}^j, \dots, v_{\sigma(u)}^j) \right] \\
 &= C \sum_{\sigma \in S_u} \text{sgn } \sigma [i(p - q)\omega \otimes \eta(\sigma v) + i(s - t)\omega \otimes \eta(\sigma v)] \\
 &= i[(p + s) - (q + t)] C \sum_{\sigma \in S_u} \text{sgn } \sigma (\omega \otimes \eta)(\sigma v) = i[(p + s) - (q + t)] \omega \wedge \eta(v),
 \end{aligned}$$

which shows that  $\omega \wedge \eta$  satisfies (4).

This result can be used to prove Theorem 1, since it is clear from Definition 1 that  $\wedge^{1,0} V' = V^*$  and  $\wedge^{0,1} V' = \overline{V^*}$ , and so by induction on Theorem 2 it follows immediately that the elements  $g_{i_1} \wedge \dots \wedge g_{i_p} \wedge \overline{g}_{j_1} \wedge \dots \wedge \overline{g}_{j_q}$  are in  $\wedge^{p,q} V'$ , where  $\{g_i : i \in I\}$  is some basis for  $V^*$ . But it is a standard fact [5, p. 7-7] that the set of all such elements (as  $p$  and  $q$  run from 0 to  $r$  subject to  $p + q = r$ ) spans  $\wedge^r V'$ . Since the subspaces  $\wedge^{p,q} V'$  are independent, the second proof of (1) is complete.

An immediate corollary of this proof is the description (2).

The third proof of Theorem 1 relies on no auxiliary results. Given  $\omega \in \wedge^r V'$ , it is desired to find functions  $\omega_p \in \wedge^{p,q} V'$  such that  $\omega = \sum_{p=0}^r \omega_p$ . By using the defining identity (4) one can obtain the following relations for the values  $\omega_p(v)$ :

$$(6.0) \quad \omega(v) = \sum_{p=0}^r \omega_p(v),$$

$$(6.1) \quad \sum_{j=1}^r \omega(v^j) = \sum_{p=0}^r \sum_{j=1}^r \omega_p(v^j) = \sum_{p=0}^r i(p - q)\omega_p(v),$$

$$(6.2) \quad \sum_{j,k=1}^r \omega((v^k)^j) = \sum_{p=0}^r i(p - q) \sum_{k=1}^r \omega_p(v^k) = \sum_{p=0}^r (i(p - q))^2 \omega_p(v),$$

⋮

$$(6t) \quad \sum_{i_1, \dots, i_t=1}^r \omega((\dots(v^{i_1}) \dots)^{i_t}) = \sum_{p=0}^r (i(p - q))^t \omega_p(v).$$

Thus (6.1) is obtained by substituting  $v^j$  for  $v$  in (6.0), summing over  $j$ , and using (4). Similarly, (6.2) follows by substituting  $v^k$  for  $v$  in (6.1), summing over  $k$ , and using (4). The first  $r + 1$  of these relations (6.0),  $\dots$ , (6. $r$ ) constitutes a linear system defined by a Vandermonde matrix whose determining entries  $\{i(p - q)\}_{p=0}^r$  are all distinct. Since the left side of each equation defines an element of  $\wedge^r V'$ , it is clear that the system has unique solutions  $\omega_0, \dots, \omega_r$  in  $\wedge^r V'$ .

It remains to show that these solutions satisfy (4), and this can be done without writing down the inverse matrix: First, one can obtain

$$0 = \sum_{p=0}^r i(p - q)\omega_p(v) - \sum_{j=1}^r \omega(v^j) = \sum_{p=0}^r \left[ i(p - q)\omega_p(v) - \sum_{j=1}^r \omega_p(v^j) \right],$$

which is the sum of the identities (4), by using  $v^j$  for  $v$  in (6.0), adding over  $j$ , and substituting the sum in (6.1). Similarly, the relation

$$\begin{aligned} 0 &= \sum_{p=0}^r (i(p - q))^2 \omega_p(v) - \sum_{k=1}^r \sum_{p=0}^r i(p - q)\omega_p(v^k) \\ &= \sum_{p=0}^r i(p - q) \left[ i(p - q)\omega_p(v) - \sum_{k=1}^r \omega_p(v^k) \right] \end{aligned}$$

is obtained by using  $v^k$  in (6.1), adding over  $k$ , and substituting the sum in (6.2). Continuing in this way, one obtains a homogeneous system of  $p + 1$  linear equations for the  $p + 1$  identities (4) with the same nonsingular Vandermonde matrix. This shows that (4) holds for each  $\omega_p$  and completes the proof.

**Remarks.** 1. If  $V$  is a topological vector space one can add continuity to the requirements defining an element of  $\wedge^r V'$ . Then Definition 1 will define independent subspaces  $\wedge^{p,q} V'$  of  $\wedge^r V'$  which are clearly weak-\* closed. Theorem 1 still holds, using its third proof (the first one also works if the elements of  $\otimes^r V'$  are required to be continuous). Therefore (1) is a topological direct sum in the weak-\* topology. Theorem 2 remains valid with the same proof. Now (2) is still true as it stands if  $\{g_i : i \in I\}$  is an algebraic basis for  $V^*$  (now the topological dual of  $V$ ), but for the usual reasons this statement is not useful for analysis. An appropriate reformulation of it will not be discussed here in any detail. One obvious case to consider is a Banach space  $V$  whose dual has a basis  $\{g_i : i = 1, 2, \dots\}$  (in the usual Banach sense). One might expect that with these interpretations the right side of (2) is, say, weak-\* dense in the left.

2. There is a more abstract way of defining  $\Lambda^{p,q}V'$ . By means of (3) and a standard construction [2, p. 121] of exterior algebra there is an isomorphism

$$T: \sum_{p+q=r} \Lambda^p V^* \otimes \Lambda^q \overline{V^*} \rightarrow \Lambda^r V',$$

where the sum is direct and  $T$  is determined on "decomposable" elements  $\omega \otimes \eta$ ,  $\omega \in \Lambda^p V^*$ ,  $\eta \in \Lambda^q \overline{V^*}$ , by  $T(\omega \otimes \eta) = \omega \wedge \eta$ . In this representation  $\Lambda^p V^*$  can be taken [2, p. 215] as the space of all alternating complex-linear maps  $\omega: V \times \cdots \times V$  ( $p$  factors)  $\rightarrow \mathbb{C}$  and  $\Lambda^q \overline{V^*}$  as the space of all alternating conjugate-linear maps  $\eta: V \times \cdots \times V$  ( $q$  factors)  $\rightarrow \mathbb{C}$ . The isomorphism can be used to define  $\Lambda^{p,q}V'$  as the image under  $T$  of  $\Lambda^p V^* \otimes \Lambda^q \overline{V^*}$ , and this is in complete agreement with Definition 1: For an element  $\omega$  of  $\Lambda^p V^*$  clearly satisfies the defining identity  $ip\omega(v) = \sum_{j=1}^p \omega(v^j)$  for  $(p, 0)$  functions, and similarly  $\Lambda^q \overline{V^*} \subset \Lambda^{0,q}V'$ . Thus Theorem 2 implies that  $T$  maps  $\Lambda^p V^* \otimes \Lambda^q \overline{V^*}$  into  $\Lambda^{p,q}V'$ . It follows from Theorem 1 that  $T$  maps onto  $\Lambda^{p,q}V'$ , so  $T$  induces an isomorphism  $\Lambda^p V^* \otimes \Lambda^q \overline{V^*} \simeq \Lambda^{p,q}V'$ .

3. Another construction which is frequently used [1, p. 22], [4, p. 495] in place of  $\Lambda^r V'$  is the exterior power  $\Lambda^r(\mathbb{C}V)^*$  of the dual of the complexification  $\mathbb{C}V$  of the real vector space underlying  $V$ .  $\Lambda^r(\mathbb{C}V)^*$  can be taken as the space of all alternating complex linear maps on  $\mathbb{C}V$ , and this interpretation leads to an alternate description of  $\Lambda^{p,q}V'$ . The reason is that  $\mathbb{C}V$  has a canonical decomposition  $\mathbb{C}V = V^{(1,0)} \oplus V^{(0,1)}$  with natural isomorphisms  $V^{(1,0)} \simeq V \simeq V^{(0,1)}$  and this induces a canonical isomorphism of  $\Lambda^r V'$  with  $\Lambda^r(\mathbb{C}V)^*$ . The latter map carries  $\Lambda^{p,q}V'$  onto the set of all  $\omega \in \Lambda^r(\mathbb{C}V)^*$  such that  $\omega(v) = 0$  if more than  $p$  of the coordinates  $v_j$  are in  $V^{(1,0)}$  or more than  $q$  of them are in  $V^{(0,1)}$  [1, p. 22], [5, p. 7-50]. In principle one could thus define  $\Lambda^{p,q}V'$  by means of this isomorphism. However it is complicated, and a proof of (4) by means of it is not well motivated. Of course one could adopt  $\Lambda^r(\mathbb{C}V)^*$  as the preferred model, and this is quite common, but it is less natural than  $\Lambda^r V'$  for some applications.

4. David Prill has pointed out to the author that  $\Lambda^{p,q}V'$  can also be characterized by the identity

$$(7) \quad e^{i\theta(p-q)}\omega(v) = \omega(e^{i\theta}v_1, \dots, e^{i\theta}v_r), \quad \text{all real } \theta,$$

and that this will yield (4) upon differentiation with respect to  $\theta$  at 0. That (7) defines  $\Lambda^{p,q}V'$  can be seen in much the same way as the third proof of Theorem 1. The special case of (7) where  $\theta = \pi/2$  occurs in [6, p. 4]. It is in fact only necessary to impose (7) for a fixed  $\theta$  such that  $e^{i\theta}$  is not a  $k$ th

root of unity,  $k \leq r$ . In any case, for fixed  $\theta$  the right side of (7) defines an operator on  $\wedge^r V'$  whose eigenspaces are the  $\wedge^{p,q} V'$  with eigenvalues  $e^{i\theta(p-q)}$ , as exhibited by (7).

(Added in proof: The identity (7) is also mentioned by M. Jambon, L'Enseignement Math. 18 (1972), 303–337, on p. 305.)

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