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THE DIFFERENTIAL EQUATION $\Delta x = 2H(x_u \wedge x_v)$ WITH VANISHING BOUNDARY VALUES

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ABSTRACT. If $x(u, v)$ is a solution to the system $\Delta x = 2H(x_u \wedge x_v)$ on a bounded domain $G \subset R^2$ with finite Dirichlet integral and with $x = 0$ on ∂G, then $x \equiv 0$ for simply connected G, but for doubly-connected G we construct nontrivial solutions.

I. Introduction. In this paper we demonstrate the following "curious" result.

Theorem. *For G a bounded domain in R^2, let $x(u, v)\colon G \to R^3$ be a vector function continuous on $\overline{G}$, twice continuously differentiable on G, whose Dirichlet integral $D(x)$ is finite where*

$$D(x) = \iint_G (|x_u|^2 + |x_v|^2)\, du\, dv = \iint_G |\nabla x|^2\, du\, dv \tag{1.1}$$

and which is a solution to the partial differential equation

$$\Delta x = 2H(x_u \wedge x_v) \tag{1.2}$$

for some constant H. If G is simply-connected (without loss of generality we may assume $G = B =$ the unit disk $u^2 + v^2 < 1$), then the only such solution with $x = 0$ on ∂G, the boundary of G, is $x \equiv 0$. On the other hand if G is doubly-connected (say an annular region), then there exist nontrivial solutions to (1.2) which vanish on ∂G for any nonzero constant H.

If $x(u, v)$ is a solution of the system (1.2) and furthermore satisfies the conformality condition

$$|x_u| = |x_v|, \qquad (x_u \cdot x_v) = 0 \tag{1.3}$$

on G, then $x(u, v)$ is a parametric representation of a surface of constant mean curvature H. In recent years many existence theorems for the system

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(1.2), (1.3) have been published, generally subject to the condition that the boundary values $x: \partial G \to R^3$ are an admissible representation of a given Jordan curve γ. Such surfaces are of interest as they give representations of soap bubbles. The first result was proven by E. Heinz [5]. For a relatively complete listing of existence proofs we refer the reader to the paper of S. Hildebrandt and H. Kaul [7].

For example S. Hildebrandt [6] proved the following theorem.

Let γ be a Jordan curve lying inside the unit ball $x^2 + y^2 + z^2 \leq 1$. Let $\mathcal{S}(\gamma)$ be the class of vector functions $x: \overline{B} \to R^3$ of class $C^0(\overline{B}) \cap C^1(B)$, with finite Dirichlet integral, and whose boundary values are a representation of γ. Then for $|H| \leq 1$, among the set of all members of $\mathcal{S}(\gamma)$ whose range is contained in the unit ball, there is at least one member x_0 which minimizes the functional

$$E_H(x) = D(x) + 4HV(x) \tag{1.4}$$

where $V(x)$ is the oriented volume functional

$$V(x) = \frac{1}{3} \iint_B x \cdot (x_u \wedge x_v)\, du\, dv. \tag{1.5}$$

This x_0 is a solution to the system (1.2), (1.3).

Note. Existence theorems of this nature have also been carried over to the case of nonconstant H. For a full discussion and list of references, the reader should consult K. Steffen [8].

If instead, one considers fixed boundary data, then one obtains solutions to (1.2) but loses the conformality (1.3). Recently, the author obtained the following existence theorem [10].

Let $y(u, v) \in C^0(\overline{B}) \cap C^1(B)$ be a vector function with finite Dirichlet integral, and denote by $\mathcal{S}(y)$ the class of functions $x \in C^0(\overline{B}) \cap C^1(B)$ with $D(x) < \infty$ and such that $x - y = 0$ on ∂B. Suppose also that for every such x, we have $A(x) > 0$ where $A(x) = \iint |x_u \wedge x_v|\, du\, dv$. Let K be a given constant. Among the set of all $x \in \mathcal{S}(y)$ such that the oriented volume enclosed by x and y, $V(x) - V(y) = K$, there is an x_0 which minimizes $D(x)$. This x_0 is a solution to the system (1.2) *for some constant H.*

One can regard x_0 as representing the shape of an ideal elastic membrane in equilibrium with fixed boundary values and subject to a pressure difference on one side.

II. The uniqueness theorem. Since the differential equation (1.2) is invariant under a conformal change of variables in the parameter domain, we

may take our simply-connected region G to be the unit disk B.

Let W_1 be the Sobolev Hilbert space of vector functions $x: B \to R^3$ such that x and its first partial derivatives x_u and x_v are square integrable. Let W_{10} be the closure in W_1 of the $C_0^\infty(B)$ functions, the C^∞ functions with compact support in B. W_{10} is a closed subspace of W_1 and on W_{10} an equivalent inner product is given by the Dirichlet inner product

$$(2.1) \qquad (x \cdot y)_D = \iint_B (x_u \cdot y_u + x_v \cdot y_v)\, du\, dv = \iint_B (\nabla x \cdot \nabla y)\, du\, dv.$$

The following result was proven in [9, p. 323].

The oriented volume functional $V(x)$ has a unique continuous extension from $C_0^\infty(B)$ to W_{10}. On W_{10}, $V(x)$ is an analytic functional and we have the isoperimetric inequality

$$(2.2) \qquad |V(x)| \leq (1/3\sqrt{32\pi})[D(x)]^{3/2}.$$

The following assertion follows directly from a regularity result proven in [9, p. 337].

Let $x \in W_{10}$ be a critical point for the functional $E_H(x)$ in the space W_{10}. This means that

$$(2.3) \qquad E'_H(x, \phi) = 2(x \cdot \phi)_D + 4HV'(x, \phi) = 0$$

for all $\phi \in W_{10}$ where $V'(x, \phi)$ is the derivative of $V(x)$ at x. Then x is analytic on B, continuous on $\bar{B}$ with $x = 0$ on ∂B and is a solution of the differential equation (1.2) on B.

We can then state the uniqueness theorem as follows.

Uniqueness theorem. *The only critical point for the functional $E_H(x)$ in $W_{10}(B)$ is $x = 0$.*

Proof. Let x be a critical point for $E_H(x)$. Then $x \in C^2(B) \cap C^0(\bar{B})$ and is a solution to (1.2) on B. We now extend the domain of x to all of R^2 by a reflection, setting

$$(2.4) \qquad y(u, v) = \begin{cases} x(u, v), & (u, v) \in \bar{B}, \\ -x(u/r^2, v/r^2), & (u, v) \notin \bar{B} \text{ where } u^2 + v^2 = r^2. \end{cases}$$

$y(u, v)$ is then a solution to (1.2) on $R^2 - \{u^2 + v^2 = 1\}$. We shall now show that $y(u, v)$ is a solution to (1.2) on all of R^2.

Note. This same assertion was shown previously by E. Heinz [5, p. 278] under slightly stronger hypotheses.

Let $\phi(u, v) \in C_0^\infty(R^2 - \{(0, 0)\})$. We write $\phi(u, v)$ as the sum of an even and an odd function relative to the unit circle $u^2 + v^2 = 1$. Set

$$\hat{\phi}(u, v) = \phi(u/r^2, v/r^2), \qquad u^2 + v^2 = r^2. \tag{2.5}$$

We can then write

$$\phi(w) = \frac{\phi(w) + \hat{\phi}(w)}{2} + \frac{\phi(w) - \hat{\phi}(w)}{2} = \phi_e(w) + \phi_0(w) \tag{2.6}$$

where $w = u + iv$. Since $y(u, v)$ is itself odd, it follows that

$$E'_H(y, \phi_e) = 2(y \cdot \phi_e)_D + 4HV'(y, \phi_e) = 0. \tag{2.7}$$

Now let $\psi = \phi_0$ restricted to $\bar{B}$. Then $\psi \in W_1(B) \cap C^0(\bar{B})$ and has zero boundary values. Therefore $\psi \in W_{10}(B)$. Since y and ϕ_0 are both odd with respect to B, we have

$$(y \cdot \phi_0)_D = 2(x \cdot \psi)_D, \qquad V'(y, \phi_0) = 2V'(x, \psi), \tag{2.8}$$

and hence

$$E'_H(y, \phi) = E'_H(y, \phi_0) + E'_H(y, \phi_e) = 2E'_H(x, \psi) + 0 = 0 \tag{2.9}$$

as x is a critical point for $E_H(x)$ on $W_{10}(B)$. By the regularity theorem it follows that $y(u, v)$ is analytic on R^2 and is a solution to (1.2). Furthermore we have $D(y) = 2D(x) < \infty$.

We now consider the "conformal measure" function

$$F(w) = (|y_u|^2 - |y_v|^2) - 2i(y_u \cdot y_v) \quad \text{(Courant [2, p. 97])}. \tag{2.10}$$

A direct computation shows that $F(w)$ is complex analytic for any solution of (1.2). Thus for $y(w)$, $F(w)$ is an entire function. But we have $|F(w)| \leq |y_u|^2 + |y_v|^2$. Therefore

$$\iint |F(w)|\, du\, dv \leq D(y) = 2D(x) < \infty.$$

The only entire function in $L_1(R^2)$ is $F(w) \equiv 0$. This means that $y(u, v)$ satisfies the conformality condition (1.3) as well.

However, by a result of P. Hartman and A. Wintner [4] (for a simple proof applicable here see [3, p. 279]) any solution to the system (1.2), (1.3) other than the constant function possesses only isolated branch points (i.e. points where $|x_u| = |x_v| = 0$). On ∂B we have $y \equiv 0$. This implies that the entire circle ∂B consists of branch points which are not isolated. Therefore $y(u, v)$ is constant, and as $x \in W_{10}(B)$ we have $y \equiv 0$. Q.E.D.

Remark. This result has the following implication as I found out in a discussion with S. Hildebrandt while visiting the University of Bonn.

On W_{10}, $x = 0$ is a strong local minimum for the functional $E_H(x)$. From the inequality (2.2) it follows that if $|x|_D = [D(x)]^{1/2} < 3\sqrt{32\pi}/8|H|$, then $E_H(x) > \frac{1}{2}D(x)$. On the other hand, if $H \neq 0$, $E_H(x)$ takes on arbitrarily large negative values. One might hope (applying a minimax argument) that there should exist unstable critical points for $E_H(x)$ on $W_{10}(B)$. Our result shows that this is not the case.

III. **Nonuniqueness on the annulus.** Since any solution to the differential equation (1.2) is invariant under a conformal transformation of the parameter domain, it follows that any solution to (1.2) on an annular region bounded by the circles $r = 1$ and $r = R_0 > 1$ and vanishing on the boundary, is equivalent to finding a solution $x(w)$ on the infinite vertical strip $0 \leq u \leq 2a$ ($2a = \ln R_0$) which is periodic in v of period 2π and vanishes on the lines $u = 0$, $u = 2a$. The conformal transformation involved is $T(w) = e^w$. We shall now construct a nontrivial solution on the infinite vertical strip.

We first note that if $x(w)$ is such a solution, then as in §II, one can extend $x(w)$ to a solution on all of R^2 by repeated reflections about the lines $u = 2na$. $x(w)$ will be doubly periodic, of period $4a$ in u and period 2π in v. Therefore, the conformal measuring function $F(w)$ (2.10) is an entire doubly periodic function and so must be a constant. If $x(w)$ is nontrivial we can immediately infer that $F(w) \neq 0$ since otherwise $x(w)$ would be conformal and the lines $u = 2na$ would be lines of branch points.

We shall construct $x(w)$ as a surface of revolution about the z-axis with the form

$$x(u, v) = \langle f(u) \cos v, f(u) \sin v, g(u)\rangle \tag{3.1}$$

with the boundary conditions

$$f(0) = g(0) = f(2a) = g(2a) = 0. \tag{3.2}$$

One straightforwardly concludes that $x(u, v)$ is a solution to the differential equation (1.2) if and only if

$$\text{(a) } f'' - f = -2Hfg', \qquad \text{(b) } g'' = 2Hff'. \tag{3.3}$$

We note here that if $H = 0$, then there are no nontrivial solutions to (3.3) periodic in u, as $f'' - f = 0$. In this case, if we also require the mapping to be conformal, solutions to (3.3) will give a representation of the catenary

$$x(u, v) = \langle \cosh u \cos v, \cosh u \sin v, u\rangle.$$

Now assume $H \neq 0$. In this case, by multiplying $x(u, v)$ by a suitable scalar factor, we may assume $H = -1$, yielding

$$\text{(3.4)} \qquad \text{(a) } f'' - f = 2fg', \qquad \text{(b) } g'' = -2ff'.$$

Upon integrating the second of these equations, we get

$$\text{(3.5)} \qquad g' = -f^2 + c$$

for some constant c. Substitute this into (3.4)(a) and we get

$$\text{(3.6)} \qquad f'' - (2c + 1)f + 2f^3 = 0.$$

This is the well-known Duffing differential equation in homogeneous form. It has the following integral:

$$\text{(3.7)} \qquad (f')^2 - (2c + 1)f^2 + f^4 = \lambda^2$$

where $\lambda^2 = [f'(0)]^2$ if $f(0) = 0$. We now apply the boundary conditions (3.2), and upon observing that (3.7) is symmetric in f and f', conclude that $f(2a) = f(a) = 0$ if f is periodic of period $4a$, and that

$$\text{(3.8)} \qquad ac = \int_0^a f^2(u)\,du$$

if $g(2a) = 0$. We obtain (3.8) by integrating (3.5) and observing that $\int_0^{2a} f^2(u)\,du = 2\int_0^a f^2(u)\,du$ by symmetry. Hence $c > 0$.

Now if f is a solution to (3.6) of period $4a$ and $f(0) = f(4a) = 0$, then an integration of (3.7) yields

$$\text{(3.9)} \qquad P(L) \equiv a = \frac{\text{Period}}{4} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{-(2c + 1) + L^2(1 + \sin^2\theta)}}$$

where L is the amplitude of f which by (3.7) satisfies

$$\text{(3.10)} \qquad L^4 - (2c + 1)L^2 = \lambda^2 = [f'(0)]^2$$

and so $\sqrt{2c + 1} < L < +\infty$ (see [1, p. 154], for example).

Given $c > 0$ we want to choose L so that (3.8) is satisfied. This means

$$\text{(3.11)} \qquad \int_0^{\pi/2} \frac{L^2 \sin^2\theta\, d\theta}{\sqrt{-(2c + 1) + L^2(1 + \sin^2\theta)}} = cP(L), \qquad (2c + 1) < L^2 < \infty,$$

or $I(L) = cP(L)$. For $P(L)$ we have

$$\text{(3.12)} \qquad \begin{aligned} &\text{(a) limit } P(L) = +\infty \quad \text{as } L \to \sqrt{2c + 1}, \\ &\text{(b) } P(L) \text{ monotonic decreasing}, \\ &\text{(c) limit } P(L) = 0 \quad \text{as } L \to +\infty. \end{aligned}$$

For $I(L)$ we have

$$(3.13)\qquad \begin{aligned} &\text{(a) } \operatorname{limit} I(L) = \sqrt{2c+1} \quad \text{as } L \to \sqrt{2c+1}, \\ &\text{(b) } \operatorname{limit} I(L) = +\infty \quad \text{as } L \to +\infty. \end{aligned}$$

Therefore, there exists at least one value $\hat{L}$, $\sqrt{2c+1} < \hat{L} < \infty$, for which (3.11) and hence (3.8) is true. For this choice of $\hat{L}$, the solution pair $f(u)$, $g(u)$ of (3.4) vanishes at $u = 0$, $u = 2a$ where $a = P(\hat{L})$. This gives us our nontrivial solution.

With a more detailed analysis of (3.11), one can show that as c varies between 0 and $+\infty$, the corresponding period $P(L)$ ranges from $+\infty$ to 0.

Finally we observe that the conformal measuring function $F(w)$ is given by the integral

$$F(w) = f'^2 + g'^2 - f^2 = \text{constant}.$$

If $f(0) = g(0) = 0$, then $f'(0) \neq 0$ if $f(u)$ is nontrivial (as f is a solution to a second order D.E.) and so $F(w) > 0$ and our constructed solution is not conformal as we had already observed must be the case.

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