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θ-REFINABILITY AND LOCAL PROPERTIES

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ABSTRACT. If Q is a property more general than metrizability, we prove several theorems of the general type: A locally Q, θ-refinable space is a Q-space.

1. **Introduction.** Let (Q) be a property for a space X. We call a space X a locally (Q)-space if each point of the space has an open neighborhood with property (Q). Smirnov [20] proved that a paracompact, locally metrizable space is metrizable. Ceder [8] proved that a paracompact, locally M_i-space is an M_i-space for $i = 1, 2, 3$. Burke [5] recently showed that a subparacompact, locally developable space is developable.

Throughout this paper, $n, m \in N$ and $\alpha \in A$. A space X is *θ-refinable* [21] if for every open cover $\mathcal{U}$ of X there is a sequence $\{\mathcal{V}_n\}$ of open refinements of $\mathcal{U}$ such that if $x \in X$, there is an $n(x) \in N$ such that x is contained in at most finitely many members of $\mathcal{V}_{n(x)}$ (i.e. $\operatorname{ord}(x, \mathcal{V}_{n(x)}) < \infty$). If $\mathcal{U} = \{U_\alpha\}$ is an open cover of X and $\{\mathcal{V}_n\}$ is a θ-refinement of $\mathcal{U}$ we may assume, without loss of generality, that $\mathcal{V}_n = \{V_n(\alpha)\}$ where $V_n(\alpha) \subseteq U_\alpha$ for each $\alpha \in A$. Such a collection $\{\mathcal{V}_n\}$ will be called an *indexed θ-refinement* of $\mathcal{U}$. Clearly every metacompact space is θ-refinable and Burke [5] proved that every subparacompact space is θ-refinable. We show in Example 4.4 that paracompactness cannot be replaced by subparacompactness, metacompactness or θ-refinability in the results of Smirnov and Ceder.

We assume all spaces are T_2. The positive integers are denoted by N.

2. **Locally semistratifiable spaces.** A space X is a *semistratifiable space* if for each open set $U \subseteq X$, there is a collection $\{U_n\}$ of closed subsets of X such that $U = \bigcup_{n=1}^{\infty} U_n$ and if $U \subseteq V$, V open, then $U_n \subseteq V_n$. The concept of a semistratifiable space is due to E. Michael and was first studied by Creede [9]. Creede proved that every semistratifiable space is subparacompact and thus θ-refinable.

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A collection $\mathcal{F}$ of closed subsets of a space X is a *ct-net* for X if for any two distinct points x, y, of X, there is an $F \in \mathcal{F}$ such that $x \in F$ and $y \notin F$. A space with a σ-closure preserving ct-net is called a *$\sigma^{\#}$-space*. These definitions were introduced by Siwiec and Nagata in [19].

A space X is a *β-space* if for each $x \in X$, there is a sequence $\{g_n(x)\}$ of open neighborhoods of x such that if $x \in g_n(x_n)$, then $\{x_n\}$ clusters. The first author and Hodel [12] independently defined β-spaces and proved Theorem 2.1.

Theorem 2.1. *A space X is a semistratifiable space if and only if X is a β-space and a $\sigma^{\#}$-space.*

The main result of this section is that a θ-refinable, locally semistratifiable space is semistratifiable. To get this we first obtain the analogous result for β-spaces and $\sigma^{\#}$-spaces and then invoke Theorem 2.1.

Theorem 2.2. *A θ-refinable, locally β-space is a β-space.*

Proof. Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of X by β-spaces. For each $x \in X$ and for each $\alpha \in A$ such that $x \in U_\alpha$, let $\{g_{n,\alpha}(x)\}$ be a sequence of open neighborhoods of x illustrating that U_α is a β-space. We may assume $g_{n+1,\alpha}(x) \subseteq g_{n,\alpha}(x)$ for all $n \in N$. Let $\{\mathcal{V}_n\}$ be an *indexed θ-refinement* of $\mathcal{U}$. For every $x \in X$ and $n \in N$ there exists an $\alpha_x \in A$ such that $x \in V_n(\alpha_x)$. Let $h_{n,m}(x) = g_{m,\alpha_x}(x) \cap V_n(\alpha_x)$ and put $h_m(x) = \bigcap_{n=1}^{m} h_{n,m}(x)$. Suppose $x_0 \in h_m(x_m)$. There exists an integer n_0 such that $\operatorname{ord}(x, \mathcal{V}_{n_0})$ is finite. For $m > n_0$, $x_0 \in h_{n_0,m}(x_m) \subseteq V_{n_0}(\alpha_{x_m})$. But $\{\alpha_{x_m} : m = 1, 2, \cdots\}$ is a finite set and hence there is an $\alpha \in A$ and a subsequence $N_1 \subseteq N - \{1, 2, \cdots, m\}$ such that $\alpha_{x_j} = \alpha$ for all $j \in N_1$. Thus $x_0 \in h_{n_0,j}(x_j) \subseteq g_{j,\alpha}(x_j)$ for all $j \in N_1$. Since U_α is a β-space, $\{x_j : j \in N_1\}$ clusters and thus the sequence $\{x_m\}$ clusters. Hence X is a β-space.

In order to establish a theorem for $\sigma^{\#}$-spaces analogous to Theorem 2.2, we need the following characterization of $\sigma^{\#}$-spaces due essentially to R. W. Heath.

Lemma 2.3. *A space X is a $\sigma^{\#}$-space if and only if for each $x \in X$, there is a sequence $\{g_n(x)\}$ of open neighborhoods of x such that $\bigcap_{n=1}^{\infty} g_n(x) = \{x\}$ and if $y \in g_n(x)$, then $g_n(y) \subseteq g_n(x)$.*

Theorem 2.4. *A θ-refinable, locally $\sigma^{\#}$-space X is a $\sigma^{\#}$-space.*

Proof. Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of X by $\sigma^{\#}$-spaces. For each $x \in X$ and for each $\alpha \in A$ such that $x \in U_\alpha$, let $\{g_{n,\alpha}(x)\}$ be a sequence of open neighborhoods of x satisfying the conditions of Lemma 2.3 for U_α. We first show that if there is a point-finite open refinement of $\mathfrak{U}$, then X is a $\sigma^\#$-space. Thus let $\mathfrak{V} = \{V_\alpha\}$ be an indexed point-finite open refinement of $\mathfrak{U}$. For each $x \in X$, let $h_n(x) = \bigcap\{g_{n,\alpha}(x) \cap V_\alpha\colon x \in V_\alpha\}$. Then it is easy to verify that $\{h_n(x)\}$ satisfies the conditions of Lemma 2.3 for X. Thus X is a $\sigma^\#$-space.

Now let $\{\mathcal{H}_n\}$ be a θ-refinement of $\mathfrak{U}$. Let $X_{n,m} = \{x \in X\colon \operatorname{ord}(x, \mathcal{H}_n) \le m\}$. Then $X_{n,m}$ is a closed subset of X and every point of $X_{n,m}$ is of finite order relative to $\mathcal{H}_n$. Since the property of being $\sigma^\#$ is hereditary, $\{U \cap X_{n,m}\colon U \in \mathfrak{U}\}$ is an open cover of $X_{n,m}$ by $\sigma^\#$-spaces. Since $\{H \cap X_{n,m}\colon H \in \mathcal{H}_n\}$ is a point-finite open refinement of $\{U \cap X_{n,m}\colon U \in \mathfrak{U}\}$, $X_{n,m}$ is a $\sigma^\#$-space. But $X = \bigcup\{X_{n,m}\}$. Since the countable union of closed $\sigma^\#$-spaces is clearly $\sigma^\#$, X is a $\sigma^\#$-space.

The following result is an immediate consequence of Theorems 2.1, 2.2 and 2.4.

Theorem 2.5. *A locally semistratifiable space X is semistratifiable if and only if X is θ-refinable.*

Creede [9] has shown that a space X is semimetrizable if and only if X is semistratifiable and first countable. Thus we have the following:

Theorem 2.6. *A locally semimetrizable space X is semimetrizable if and only if X is θ-refinable.*

A collection $\mathfrak{B}$ of subsets of a space X is called a *network* for X if for any open set $U \subseteq X$ and $x \in U$ there is a set $B \in \mathfrak{B}$ such that $x \in B \subseteq U$. A space with a σ-locally finite closed network is called a *σ-space* [18] (see also [19]). It is easy to verify that every σ-space is semistratifiable.

Theorem 2.7. *A locally σ-space is a σ-space if and only if X is θ-refinable.*

Proof. The necessity is obvious. Conversely, let $\mathfrak{U} = \{U_\alpha\}$ be an open cover of X by σ-spaces. Since each U_α is semistratifiable, it follows from Theorem 2.5 that X is semistratifiable and hence subparacompact. Let $\mathcal{F} = \bigcup_{n=1}^\infty \mathcal{F}_n$ be a σ-discrete closed refinement of $\mathfrak{U}$. Since the property of σ is hereditary, X has a σ-discrete closed cover by σ-spaces. Hence X is a σ-space.

A class of spaces which simultaneously generalizes σ-spaces and M^*-spaces [13] is the class of Σ-spaces introduced by Nagami [16]. A space X is a *Σ-space* if there is a sequence of locally finite closed covers $\{\mathcal{F}_n\}$ of X such that if $x_n \in \bigcap\{F \in \mathcal{F}_n\colon x \in F\}$ for some fixed point $x \in X$, then

$\{x_n\}$ clusters. Michael [14] has pointed out that replacing "σ-locally finite" by "σ-closure preserving" in the definition of Σ-space leads to a strictly larger class of spaces, which are called *$\Sigma^{\#}$-spaces.*

It is unknown if a θ-refinable, locally Σ-space is a Σ-space. In fact, it is not even known if the union of two open Σ-spaces is a Σ-space. However, by Corollary 1.10 and Theorem 3.2 of [16], we have the following partial result.

Theorem 2.8. *A subparacompact, locally Σ-space is a Σ-space.*

On the other hand, using a characterization of $\Sigma^{\#}$-spaces given by Nagata [17] (see also [2]) we can obtain the following theorem. The proof is essentially the same as the proof of Theorems 2.2 and 2.4 and is omitted.

Theorem 2.9. *A θ-refinable, locally $\Sigma^{\#}$-space is a $\Sigma^{\#}$-space.*

3. **Locally p-spaces and locally $w\Delta$-spaces.** By Arhangel'skiĭ [1], a completely regular space X is called a *p-space* if there is a sequence $\{\mathfrak{U}_n\}$ of open (in βX) covers of X such that if $x \in X$, $\bigcap_{n=1}^{\infty} \operatorname{St}(x, \mathfrak{U}_n) \subseteq X$. If, in addition, for each $x \in X$ and $n \in N$ there exists an $n(x) \in N$ such that $\overline{\operatorname{St}(x, \mathfrak{U}_{n(x)})} \subseteq \operatorname{St}(x, \mathfrak{U}_n)$, then X is called a *strict p-space.*

A space X is a *$w\Delta$-space* [4] if there is a sequence $\{\mathfrak{U}_n\}$ of open covers of X such that if $x_n \in \operatorname{St}(x, \mathfrak{U}_n)$, then $\{x_n\}$ clusters.

Creede [9] introduced the class of quasi-complete spaces which simultaneously generalizes p-spaces and $w\Delta$-spaces. A space X is a *quasi-complete space* if there is a sequence $\{\mathfrak{U}_n\}$ of open covers of X such that if $\{x_k\colon k > n\} \cup \{x\} \subseteq U \in \mathfrak{U}_n$ for some fixed point $x \in X$, then $\{x_n\}$ clusters.

In order to obtain the results of this section we need the following lemma.

Lemma 3.1 (Burke [6]). *For a completely regular θ-refinable space X, the following are equivalent:*

(a) *X is a p-space.*

(b) *X is a strict p-space.*

(c) *X is a $w\Delta$-space.*

(d) *X is a quasi-complete space.*

Moreover, conditions (c) *and* (d) *are equivalent for any θ-refinable space X.*

It should be noted that the equivalence of (a), (b) and (c) is the content of Theorem 1.7 of [6]. The "moreover" is Corollary 3.1.8 of [10].

Theorem 3.2. *A θ-refinable, locally quasi-complete space X is quasi-complete.*

Proof. Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of X by quasi-complete spaces. For each $\alpha \in A$, let $\{\mathcal{G}_{n,\alpha}\}$ be a sequence of open covers of U_α illustrating that U_α is quasi-complete. We may assume $\mathcal{G}_{n+1,\alpha} < \mathcal{G}_{n,\alpha}$ for all $n \in N$. Let $\{\mathcal{V}_n\}$ be an indexed θ-refinement of $\mathcal{U}$. Let us put $\mathcal{W}_{n,m} = \{G \cap V_n(\alpha)$: $G \in \mathcal{G}_{m,\alpha}\}$ and $\mathcal{H}_k = \bigwedge_{n+m=2}^{k+1} W_{n,m}$. Then for each $k \in N$, $\mathcal{H}_k$ is an open cover of X.

Suppose $\{x_i: i > n\} \cup \{x\} \subset H_n \in \mathcal{H}_n$ for some fixed point $x \in X$. There exists an integer n_0 such that $\operatorname{ord}(x, \mathcal{V}_{n_0})$ is finite. For each $k \in N$, put $S_k = \{x_i: i \geq n_0 + k\}$. Since $S_k \subseteq H_{n_0+k} \in \mathcal{H}_{n_0}$ it follows that $S_k \subseteq W$ for some $W \in \mathcal{W}_{n_0,k}$. Thus there exists an $\alpha_k \in A$ and a $G_k \in \mathcal{G}_{k,\alpha_k}$ such that $S_k \subseteq G_k \cap V_{n_0}(\alpha_k)$. But $\{\alpha_k: k \in N\}$ is a finite set and hence there is an $\alpha \in A$ and a subsequence $N_1 \subseteq N$ such that $\alpha_j = \alpha$ for all $j \in N_1$. Since $S_j = \{x_i: i \geq n_0 + j\} \cup \{x\} \subseteq G_j \in \mathcal{G}_{j,\alpha}$, the sequence $\{x_n\}$ clusters.

The next two results are an immediate consequence of Lemma 3.1 and Theorem 3.2.

Theorem 3.3. *A θ-refinable, locally $w\Delta$-space is a $w\Delta$-space.*

Theorem 3.4. *A completely regular θ-refinable locally p-space (locally strict p-space) is a p-space (strict p-space).*

4. Applications and examples.

Theorem 4.1. *A locally Moore space X is a Moore space if and only if X is θ-refinable.*

Proof. A locally Moore space is both locally semistratifiable and locally quasi-complete. Thus, by Theorems 3.2 and 2.5, X is both semistratifiable and quasi-complete. It follows from [9, Theorem 4.6] that X is a Moore space.

We note that Theorem 4.1 generalizes, at least for regular spaces, the result of Burke mentioned in the Introduction.

Theorem 4.2 (Smirnov [20]). *A locally metrizable space X is metrizable if and only if X is paracompact.*

Proof. A locally metrizable space is a locally Moore space and hence a Moore space by Theorem 4.1. But a paracompact Moore space is metrizable [3].

The next result follows immediately from Corollary 3.5. For the appropriate definitions the reader is referred to [15].

Theorem 4.3. *A paracompact, locally M (M^*, $M^\#$, or wM)-space X is an M (M^*, $M^\#$, or wM)-space.*

Example 4.4. Let S be the space of Example 1 in [11]. The space S is a metacompact Moore space which is locally metrizable, but not metrizable. Thus paracompactness in Theorems 4.2 and 4.3 cannot be replaced by metacompactness, subparacompactness or θ-refinability.

Example 4.5. Let X be the space constructed by Burke in [7]. This is an example of a locally compact, locally metrizable space which is not θ-refinable. This example shows that θ-refinability is necessary in Theorems 2.2, 2.5–2.7, 2.9, 3.3, and subparacompactness is necessary in Theorem 2.8.

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