PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 50, July 1975

THE GE_2 PROPERTY FOR DISCRETE SUBRINGS OF C

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ABSTRACT. The collection of discrete subrings of the complex numbers whose SL_2 is generated by elementary matrices is completely determined.

In this note it is shown that in the collection of discrete subrings of the complex numbers, precisely seven are GE_2-rings. This generalizes a theorem of Cohn [C, Theorem 6.1, p. 22] to include those discrete subrings which are not integrally closed. I would like to thank K. Brown, G. Cooke and S. Geller for a number of enlightening discussions.

A ring R is *discretely normed* [C, p. 16] if there is a function $|\ |: R \to \mathbf{R}$ which satisfies

N1. $|x| \geq 0$ with equality if and only if $x = 0$,

N2. $|x + y| \leq |x| + |y|$,

N3. $|xy| = |x|\,|y|$,

N4. $|x| \geq 1$ for all $x \neq 0$ with equality only if $x \in R^*$,

N5. there exists no $x \in R$ such that $1 < |x| < 2$.

We now seek to determine which subrings of $\mathbf{C}$ are discretely normed by the usual absolute value function. By N4 such a subring must be discrete (i.e., the subring has at most finitely many points in common with any given compact subset of $\mathbf{C}$).

Proposition 1. *A subring of* $\mathbf{C}$ *is discrete if and only if it is contained in the ring of integers of a quadratic imaginary extension of* $\mathbf{Q}$.

Since such rings are clearly discrete, we need only show that any discrete subring of $\mathbf{C}$ is contained in the ring of integers of a quadratic imaginary extension of $\mathbf{Q}$. The additive group of such a ring R must be a discrete subgroup of $\mathbf{C}$. Hence its rank as a $\mathbf{Z}$-module is less than or equal to

Received by the editors May 29, 1974.

AMS (MOS) subject classifications (1970). Primary 20H25, 20G30, 20G35, 18F25.

Key words and phrases. GE_n-rings, general linear group, special linear group, elementary group.

2 [S, Theorem 1, p. 53]. If $\gamma \in R$, then $\gamma R \subset R$ and hence γ is integral over $\mathbf{Z}$ since R is a finitely generated $\mathbf{Z}$-module. As the $\mathbf{Z}$-rank of R is no larger than 2, the field of fractions of R is an extension of degree no larger than 2 over $\mathbf{Q}$. If $R \neq \mathbf{Z}$, the extension in question must be imaginary as the real numbers contain no discrete subring other than $\mathbf{Z}$.

Next we determine which of these rings are discretely normed by $| \ |$.

Proposition 2. *All of the discrete subrings of* $\mathbf{C}$ *are discretely normed by* $| \ |$ *except for* (i) *the full rings of integers in* $\mathbf{Q}(\sqrt{-d})$, $d = 1, 2, 3, 7, 11$, *and* (ii) $\mathbf{Z}[\sqrt{-3}]$.

Let $\{1, \omega\}$ be the usual basis for the ring of integers in a quadratic extension of $\mathbf{Q}$ [S, p. 35]. Any discrete subring of $\mathbf{C}$ can be described as $\mathbf{Z}[m\omega]$, $m \geq 0$, for some such ω. Cohn [C, p. 21] observes that the full rings of integers in quadratic imaginary number fields are discretely normed by $| \ |$ except for those in $\mathbf{Q}(\sqrt{-d})$, $d = 1, 2, 3, 7, 11$. As any subring of a discretely normed ring is also discretely normed, we need only consider the subrings of the 5 exceptions. By an easy case-by-case computation we see that N5 is assured if $m \geq 2$ for $d = 1, 2, 7, 11$ and $m \geq 3$ for $d = 3$. As $1 < |\sqrt{-3}| < 2$, $\mathbf{Z}[\sqrt{-3}]$ is not discretely normed.

According to Cohn a ring R is a GE_n-ring if the group of all $n \times n$ invertible matrices, $\mathrm{GL}_n(R)$, is generated by the elementary matrices $E_{ij}(r)$, $i \neq j$, together with the diagonal matrices. The main result is the following

Theorem 3. *A discrete subring of* $\mathbf{C}$ *is a* GE_2*-ring if and only if it is one of the following* 7 *rings*:

(1) *the full ring of integers in* $\mathbf{Q}(\sqrt{-d})$, $d = 1, 2, 3, 7, 11$,

(2) $\mathbf{Z}$,

(3) $\mathbf{Z}[\sqrt{-3}]$.

The major part of the proof follows from a result of Cohn:

Proposition 4. *Let* R *be a subring of* $\mathbf{C}$ *which is discretely normed by* $| \ |$ *and whose only units are* ± 1. *If there exist elements* $a, b \in R$ *such that*

(i) $|a| = |b|$,

(ii) $|a \pm b| \geq |b|$,

(iii) $aR + bR = R$,

then R *is not a* GE_2*-ring.*

This result is the major step in the proof of [C, Theorem 6.1, p. 22].

Corollary 5. *If R is a discrete subring of* $\mathbf{C}$ *which is not one of the 7 exceptions listed above, then R is not a GE_2-ring.*

The rings in question have only ± 1 as units and according to Cohn's result we need only exhibit appropriate elements $a, b \in R$.

As observed earlier any such ring is of the form $\mathbf{Z}[m\omega]$, $m \neq 0$. Let p be a rational prime which splits in $\mathbf{Z}[\omega]$. If t denotes the order of the class group of $\mathbf{Z}[\omega]$, we obtain $(p)^t = \mathfrak{p}_1^t \mathfrak{p}_2^t = (a_0)(b_0)$ for some $a_0, b_0 \in \mathbf{Z}[\omega]$ which are both relatively prime and conjugate (hence $|a_0| = |b_0| = p^{t/2}$). Thus for any $k \geq 1$, a_0^k, b_0^k will satisfy both (i) and (iii) of Proposition 4. We must now choose p and k so that (ii) holds and so that a_0^k, b_0^k lie in the subring R.

Now let $N = |m\omega|^2$ and using quadratic reciprocity choose an integer M_ω so that $p \equiv 1 \bmod M_\omega$ implies that p splits in $\mathbf{Z}[\omega]$ (see [S, pp. 76–80]).

By the Dirichlet theorem on arithmetic progressions, there exists a rational prime p such that $p \equiv 1 \bmod NM_\omega$. Thus p splits in $\mathbf{Z}[\omega]$ and p is relatively prime to $m\omega$. As $a_0 b_0 = \zeta p^t$ for some 6th root of unity ζ in $\mathbf{Z}[\omega]$, we have $p^{6t} = a_0^6 b_0^6 \equiv 1 \bmod m\omega$. The ring $\mathbf{Z}[\omega]/(m\omega)$ is finite and, hence, there exists an integer $s \geq 1$ such that $a_0^s \equiv b_0^s \equiv 1 \bmod m\omega$. We now let $a_1 = a_0^s$, $b_1 = b_0^s \in \mathbf{Z}[m\omega]$. $\mathbf{Z}[\omega]$ is integral over $\mathbf{Z}[m\omega]$ and hence a pair of elements in $\mathbf{Z}[m\omega]$ are relatively prime if and only if they are relatively prime in $\mathbf{Z}[\omega]$. This follows as any maximal ideal of $\mathbf{Z}[m\omega]$ has a maximal ideal of $\mathbf{Z}[\omega]$ lying over it.

Let $\alpha = a_1/b_1$. If we find an integer r such that $|\alpha^r \pm 1| \geq 1$, then (ii) will be satisfied by $a = a_1^r$, $b = b_1^r$. Since α is not a root of unity (a_1 and b_1 are relatively prime) and $|\alpha| = 1$, the numbers α^l are dense in the unit circle. We can thus choose an r so that α^r is sufficiently close to $\sqrt{-1}$ to insure that $|\alpha^r \pm 1| \geq 1$ (cf. [C, p. 22]). This completes the proof of the corollary.

The first 6 rings listed in Theorem 3 are euclidean and hence GE_n-rings for all $n \geq 2$. We now show that the last ring, $\mathbf{Z}[\sqrt{-3}]$, is also a GE_n-ring for $n \geq 2$ by generalizing a theorem of Keating [K, Proposition 1.2].

Proposition 6. *Let $R \subset S$ be rings and assume*

(1) *$S^* \cap R = R^*$ and any one-sided unit of S is a unit,*

(2) *there exists a function ϕ: $R \to W$ (W a well-ordered set) such that for every $a, b \in R$, $b \neq 0$, either*

(i) *there exist $q, r \in R$, $r \neq 0$ such that $a = qb + r$ with $\phi(r) < \phi(b)$, or*

(ii) *there exists an* $s \in S$ *such that* $a = sb$.

Then R *is a* GE_n*-ring for all* $n \geq 2$.

For $M = (a_{ij}) \in \mathrm{GL}_n(R)$ define $\phi(M) = \min\{\phi(a_{ij}) | a_{ij} \neq 0\}$. The proof will be by induction on n and $\phi(M)$: induction on $\phi(M)$ will be used for the case $n = 2$ and for the induction step from $n - 1$ to n.

Let $M \in \mathrm{GL}_n(R)$ and let a_{kl} be such that $\phi(M) = \phi(a_{kl})$. Note that any permutation matrix in $\mathrm{GL}_n(R)$ is a product of elementary and diagonal matrices. We can find two permutation matrices $P_1, P_2 \in \mathrm{GL}_n(R)$ so that $P_1 M P_2$ has a_{kl} in the (n, n)-position. Clearly $\phi(P_1 M P_2) = \phi(M)$. Thus we may assume that $\phi(M) = \phi(a_{nn})$. If $a_{nn} \in R^*$, we can put zeros in the off-diagonal positions of the nth row and column by multiplying M on the right and left by appropriate elementary matrices. Upon multiplying by an invertible diagonal matrix with a_{nn}^{-1} in the (n, n)-position we reduce to a matrix in $\mathrm{GL}_{n-1}(R)$. In case $n = 2$, this would be a scalar matrix and we would be done.

We now assume that a_{nn} is not a unit. If there exists an a_{in}, $i \neq n$, and $q, r \in R$, $r \neq 0$, such that $a_{in} = q a_{nn} + r$ with $\phi(r) < \phi(a_{nn})$, then multiplication of M on the left by $E_{in}(-q)$ puts an r in the (i, n)-position and reduces the ϕ value. If no such a_{in} exists, then by (ii) there exist $s_i \in S$, $1 \leq i \leq n - 1$, so that $a_{in} = s_i a_{nn}$. Hence in $\mathrm{GL}_n(S)$ we may factor M into a product of two matrices, the second being diagonal with all 1's except for a_{nn} in the (n, n)-position. Since this matrix is invertible, $a_{nn} \in S^*$ and hence by (1), $a_{nn} \in R^*$. As a_{nn} was not a unit by hypothesis, this case does not occur and the proof is complete.

We now verify the hypotheses of Proposition 6 for $R = \mathbf{Z}[\sqrt{-3}]$, $S = \mathbf{Z}[\omega]$, $\omega = (1 + \sqrt{-3})/2$, and $\phi = |\ |$ the absolute value function on $S \subset \mathbf{C}$.

Lemma 7. *Let* $R \subset S \subset \mathbf{C}$ *be rings which are integral over* $\mathbf{Z}$. *Then* $S^* \cap R = R^*$.

If $\alpha \in S^* \cap R$, then $N_{\mathbf{Q}(\alpha)/\mathbf{Q}}(\alpha) = \pm 1$ and α has characteristic polynomial of the form $X^n + \cdots + a_1 X \pm 1 \in \mathbf{Z}[X]$ where $[\mathbf{Q}(\alpha) : \mathbf{Q}] = n$. Thus $\alpha \in R^*$ as its inverse is a polynomial in α with coefficients in $\mathbf{Z}$.

Lemma 8 (cf. [K, Proposition 1.1]). *For any pair of elements* $a, b \in \mathbf{Z}[\sqrt{-3}]$, $b \neq 0$, *either*

(1) *there exist* $q, r \in \mathbf{Z}[\sqrt{-3}]$ *such that* $a = qb + r$ *with* $|r| < |b|$, *or*

(2) *there exists* $q \in \mathbf{Z}[\sqrt{-3}]$ *such that* $a = (q + \omega)b$.

As in one of the standard proofs that $\mathbf{Z}[\omega]$ is euclidean, we consider

$ab^{-1} = c_0 + c_1\sqrt{-3} \in \mathbf{Q}(\sqrt{-3})$ and write $c_i = [c_i] + s_i,\ 0 \le s_i < 1$. If we now let

$$q_i = \begin{cases} [c_i] & \text{if } s_i \le \tfrac{1}{2}, \\ [c_i] + 1 & \text{if } s_i > \tfrac{1}{2}, \end{cases}$$

then $(c_i - q_i)^2 \le \tfrac{1}{4}$ and hence $|ab^{-1} - q| \le \sqrt{\tfrac{1}{4} + \tfrac{1}{4} \cdot 3} = 1$, with equality only if $s_0 = s_1 = \tfrac{1}{2}$. In case strict inequality holds we have $|a - qb| < |b|$ yielding (1) for $r = a - qb$. If equality holds, we have $ab^{-1} - q = \omega$ or $a = (q + \omega)b$ yielding (2).

Remarks. 1. G. Cooke has observed that the techniques of [Co] can be used to exhibit other examples of nonintegrally closed rings of integers which satisfy Proposition 6. Taking $R = \mathbf{Z}[2\sqrt{2}]$, $S = \mathbf{Z}[\sqrt{2}]$ and $\phi(a)$ to be the absolute value of the norm $N_{\mathbf{Q}(\sqrt{2})/\mathbf{Q}}(a)$ gives such an example.

2. In case R is a Dedekind ring of arithmetic type, Bass, Milnor and Serre [B-M-S, p. 95] have shown that R is a GE_n-ring for $n \ge 3$. If R^* is infinite, Vaseršteĭn [V] has shown that R is also a GE_2-ring. The case where R^* is finite is also known by results of Cohn [C] and Serre [Se].

Now let R be an integral domain with field of fractions F where F is either algebraic over $\mathbf{Q}$ or F has characteristic $p > 0$ and has transcendence degree 1 over the prime field. Swan [Sw, Theorem 5.1] has shown that R is a GE_n-ring for $n \ge 3$ except possibly when F is totally imaginary of characteristic zero and R is integral over $\mathbf{Z}$. In this case we show that true exceptions exist, thus answering a question of Swan [Sw, Remark at the end of §5].

Let p be an odd prime and let ζ be a primitive pth root of unity. Let I be the principal ideal of $\mathbf{Z}[\zeta]$ generated by $(1 - \zeta)^2$. We let A denote the ring $\mathbf{Z} + I$.

Proposition 8. *There exists a surjective homomorphism from* $SK_1(A)$ *to the cyclic group of order* p. *In particular,* A *is not a* GE_n*-ring for any* $n \ge 2$.

We apply the K-theory exact sequence of an ideal [Mi, Theorem 6.2, p. 54] to obtain the exact sequence

$$K_2(A) \longrightarrow K_2(A/I) \longrightarrow SK_1(A, I) \longrightarrow SK_1(A) \longrightarrow 1.$$

Now $A/I \approx \mathbf{Z}/\mathbf{Z} \cap I = \mathbf{Z}/p\mathbf{Z}$ and as $K_2(\mathbf{Z}/p\mathbf{Z}) = 1$ [Mi, Corollary 9.9, p. 75], we have $SK_1(A, I) \approx SK_1(A)$. Thus the first result follows from the corresponding result for $SK_1(A, I)$ proved by Swan [Sw, Lemma 3.2].

For R a commutative ring, $\mathrm{SL}(n, R)$ denotes the group of $n \times n$ matrices of determinant 1 and $E(n, R)$ denotes the subgroup generated by the elementary matrices. It is easy to see that R is a GE_n-ring if and only if $\mathrm{SL}(n, R) = E(n, R)$. Now as A has Krull dimension 1, $\mathrm{SL}(n, A)/E(n, A) \approx SK_1(A)$ for all $n \geq 3$, and $\mathrm{SL}(2, A)$ surjects onto $SK_1(A)$ [**B**, p. 241]. Thus the last statement of the proposition is an immediate consequence of the first.

REFERENCES

[B] H. Bass, *Algebraic K-theory*, Benjamin, New York, 1968. MR 40 #2736.

[B-M-S] H. Bass, J. Milnor and J.-P. Serre, *Solution of the congruence subgroup problem for* SL_n $(n \geq 3)$ *and* Sp_{2n} $(n \geq 2)$, Inst. Hautes Études Sci. Publ. Math. No. 33 (1967), 59–137. MR 39 #5574.

[C] P. M. Cohn, *On the structure of the* GL_2 *of a ring*, Inst. Hautes Études Sci. Publ. Math. No. 30 (1966), 5–53. MR 34 #7670.

[Co] G. Cooke, *A weakening of the Euclidean property for integral domains and applications to algebraic number theory* (to appear).

[K] M. E. Keating, *On the K-theory of the quaternion group*, Mathematika 20 (1973), 59–62.

[Mi] J. W. Milnor, *Introduction to algebraic K-theory*, Ann. of Math. Studies, no. 72, Princeton Univ. Press, Princeton, N. J., 1971.

[S] P. Samuel, *Algebraic theory of numbers*, Houghton Mifflin, Boston, Mass., 1970. MR 42 #177.

[Se] J.-P. Serre, *Le problème des groupes de congruence pour* SL_2, Ann. of Math. (2) 92 (1970), 489–527. MR 42 #7671.

[Sw] R. G. Swan, *Excision in algebraic K-theory*, J. Pure Appl. Algebra 1 (1971), 221–252. MR 46 #7354.

[V] L. N. Vaseršteĭn, *On the group* SL_2 *over Dedekind rings of arithmetic type*, Mat. Sb. 89 (131) (1972), 313–322; English transl., Amer. Math. Soc. Transl., 18 (1972), 321–332.

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