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CRITERIA FOR ABSOLUTE CONVERGENCE OF FOURIER SERIES

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ABSTRACT. Let $f \in L^1(\mathbf{T})$. Define f_α by $f_\alpha(x) = f(x + \alpha)$. Then the Fourier series of f is absolutely convergent if and only if there exists a Lebesgue point α for f such that both sequences $\langle(\mathrm{R}_e\, \hat{f}_\alpha(n))^-\rangle_{n \in \mathbf{Z}}$ $\langle(\mathfrak{I}_m\, \hat{f}_\alpha(n))^-\rangle_{n \in \mathbf{Z}}$ belong to l^1. The theorem remains true if the sentence "there exists a Lebesgue point α for f" is replaced by "there is $\alpha \in \mathbf{R}$ such that f is essentially bounded in some neighborhood of α".

1. **Introduction.** One of the primary objectives in the theory of Fourier series is the study of the class $\mathfrak{A}$, that is the class of all Lebesgue integrable complex-valued functions on the circle $\mathbf{T}$ (the additive group of the reals modulo 2π) whose Fourier series are absolutely convergent. We denote by $\mathbf{A}$ the collection of all continuous complex-valued functions on $\mathbf{T}$ with absolutely convergent Fourier series. Clearly $\mathbf{A} \subset \mathfrak{A}$, and every function in $\mathfrak{A}$ is equal almost everywhere to a function in $\mathbf{A}$. An approach to studying $\mathfrak{A}$ or $\mathbf{A}$ (for which the reader is referred to [1]) has concentrated attention on seeking conditions on a function f, which are sufficient or necessary, that ensure that $f \in \mathfrak{A}$. It seems that there is no complete solution to the problem of characterizing the elements of $\mathfrak{A}$ directly in terms of their functional values. The following theorem is well known [1, p. 9]: *Every continuous function on* $\mathbf{T}$ *with nonnegative Fourier coefficients belongs to* $\mathbf{A}$. *Every function in* $\mathbf{A}$ *is a linear combination of continuous functions on* $\mathbf{T}$ *with nonnegative Fourier coefficients.*

In this paper we first give an easy generalization of this theorem by requiring that the Fourier coefficients be confined in a certain region of the complex plane instead of lying all on the positive x-axis. This is the content of Theorem 1. Next we prove our main results given by Theorems 2 and 4 of §3. Each of these theorems provides a necessary and sufficient condition for a function to be in the class $\mathfrak{A}$.

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2. **Notation.** $\mathbf{R}$ is the real line. $\mathbf{L}^1(\mathbf{T})$ is the space of all Lebesgue integrable complex-valued functions on $\mathbf{T}$. For each $f \in \mathbf{L}^1(\mathbf{T})$ the numbers

$$\hat{f}(n) = \frac{1}{2\pi} \int_{\mathbf{T}} f(t) e^{-int}\, dt \qquad (n \in \mathbf{Z})$$

are the Fourier coefficients of f, where $\mathbf{Z}$ is the group of integers. As usual we call $\Sigma_{n \in \mathbf{Z}} \hat{f}(n) e^{int}$ the Fourier series of f. Let f be Lebesgue integrable on an interval $[a, b] \subset \mathbf{R}$ and let $x \in (a, b)$. Then x is called a *Lebesgue point* for f if and only if

$$\lim_{h \to 0+} \frac{1}{h} \int_0^h |f(x+t) + f(x-t) - 2f(x)|\, dt = 0.$$

The set of all Lebesgue points for f is called the *Lebesgue set* for f. If $f \in \mathbf{L}^1(\mathbf{T})$ then the complement of the Lebesgue set for f is of Lebesgue measure zero. If f is continuous at x, then x is a Lebesgue point for f. The space of all infinite sequences of complex numbers $\langle c_n \rangle$, such that $\Sigma_{n \in \mathbf{Z}} |c_n| < +\infty$, is denoted by l^1.

For $a \in \mathbf{R}$, f_a is defined by $f_a(t) = f(t+a)$, $a^+ = \max(a, 0)$ and $a^- = \max(-a, 0)$.

As usual $\mathrm{R_e}\, z$ and $\mathfrak{I}_{\mathrm{m}}\, z$ mean the real and imaginary parts of z respectively.

3. **The main theorems.**

Theorem 1. *Let f be a continuous complex-valued function on $\mathbf{T}$ with the property: there is $\alpha \in \mathbf{R}$ such that $\alpha \leq \arg \hat{f}(n) \leq \alpha + (\pi/2)$ $(n \in \mathbf{Z})$. Then $f \in \mathbf{A}$. Also every $f \in \mathbf{A}$ is a linear combination of continuous functions on $\mathbf{T}$ with the above property.*

Proof. The second part of the theorem is obvious. To prove the first part, observe that we may assume $\alpha = 0$. For if $\alpha \neq 0$ then we consider the function $g(x) = f(x)e^{-i\alpha}$ for which $0 \leq \arg \hat{g}(n) \leq \pi/2$. Next set

$$F(x) = \frac{f(x) + \overline{f(-x)}}{2}, \qquad G(x) = \frac{f(x) - \overline{f(-x)}}{2i}.$$

Clearly both F and G are continuous. Also

$$\hat{F}(n) = \mathrm{R_e}\, \hat{f}(n) \geq 0, \qquad \hat{G}(n) = \mathfrak{I}_{\mathrm{m}}\, \hat{f}(n) \geq 0.$$

It follows from the theorem stated in the introduction that $F \in \mathbf{A}$, $G \in \mathbf{A}$ so that $f \in \mathbf{A}$. □

Theorem 2. *Let $f \in \mathbf{L}^1(\mathbf{T})$. Then $f \in \mathfrak{A}$ if and only if the following condition is satisfied:*

(*) *"there is a Lebesgue point α for f such that both sequences $\langle(\mathrm{R_e}\,\hat{f}_\alpha(n))^-\rangle_{n\in\mathbf{Z}}$, $\langle(\mathfrak{I}_\mathrm{m}\,\hat{f}_\alpha(n))^-\rangle_{n\in\mathbf{Z}}$ belong to l^1."*

Proof. Suppose (*) is satisfied. We first consider the case $\alpha = 0$. For N a positive integer, set

$$\sigma_N(t) = \sum_{n=-N}^{N} (1 - |n|/N)\,\hat{f}(n)e^{int}.$$

By a theorem of Lebesgue we know that if t is a Lebesgue point for f then $\lim_{N\to\infty} \sigma_N(t) = f(t)$. Hence

$$\lim_{N\to\infty} \sigma_N(0) = \lim_{N\to\infty} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right)\hat{f}(n) = f(0) = \text{finite}. \tag{1}$$

Also

$$\begin{aligned}\sigma_N(0) &= \sum_{n=-N}^{N} (1 - |n|/N)\hat{f}(n)\\ &= \sum_{n=-N}^{N} (1 - |n|/N)(\mathrm{R_e}\,\hat{f}(n))^+ + i\sum_{n=-N}^{N} (1 - |n|/N)(\mathfrak{I}_\mathrm{m}\,\hat{f}(n))^+ \\ &\quad - \sum_{n=-N}^{N} (1 - |n|/N)(\mathrm{R_e}\,\hat{f}(n))^- - i\sum_{n=-N}^{N} (1 - |n|/N)(\mathfrak{I}_\mathrm{m}\,\hat{f}(n))^-.\end{aligned} \tag{2}$$

If we let $N \to \infty$ then the $\sigma_N(0)$ are uniformly bounded because of (1) while the last two sums of the right-hand side of (2) are bounded (more precisely they converge) because of the hypothesis (*). Therefore,

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right)(\mathrm{R_e}\,\hat{f}(n))^+ < +\infty, \quad \lim_{N\to\infty} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right)(\mathfrak{I}_\mathrm{m}\,\hat{f}(n))^+ < +\infty.$$

Since the Cesàro summability of a series with nonnegative terms implies the convergence of the series, it follows that $\Sigma_{n\in\mathbf{Z}} |\hat{f}(n)| < +\infty$: i.e., $f \in \mathfrak{A}$.

Next assume $\alpha \neq 0$. Then 0 is a Lebesgue point for f_α, so that by the result that we just proved we have $\Sigma_{n\in\mathbf{Z}} |\hat{f}_\alpha(n)| < +\infty$. Since $\hat{f}_\alpha(n) = e^{i\alpha n}\hat{f}(n)$ we have $f \in \mathfrak{A}$. Now suppose $f \in \mathfrak{A}$ and let α be any Lebesgue point for f. Then

$$\sum_{n\in\mathbf{Z}} |\hat{f}(n)| = \sum_{n\in\mathbf{Z}} |e^{i\alpha n}\hat{f}(n)| = \sum_{n\in\mathbf{Z}} |\hat{f}_\alpha(n)| < +\infty$$

and condition (*) is clearly satisfied. □

Corollary 3. *Let $f \in \mathrm{L}^1(\mathbf{T})$. Then f is equal almost everywhere to a*

linear combination of positive definite functions if and only if condition (*) *is satisfied.*

Proof. This follows from Kahane's characterization of A (see §1) and Herglotz's characterization of continuous functions with nonnegative coefficients as positive definite [2, p. 19].

Theorem 4. *Let* $f \in L^1(T)$. *Then* $f \in \mathfrak{A}$ *if and only if the following condition is satisfied*

(**) "*f is essentially bounded in a neighborhood of some real number* α, *and both sequences* $\langle (R_e f_\alpha(n))^- \rangle_{n \in Z}$, $\langle (\mathfrak{I}_m f_\alpha(n))^- \rangle_{n \in Z}$ *belong to* l^1".

Proof. Suppose that condition (**) is satisfied. We first consider the case $\alpha = 0$. Using the notation of Theorem 2 we have:

$$\sigma_N(t) = \frac{1}{2\pi} \int_T f(y) K_N(t-y)\, dy$$

where

$$K_N(y) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{iny} = \frac{\sin^2 (N/2)y}{N \sin^2 (y/2)}.$$

Next assume $|f(y)| \leq M$ a.e. for $y \in (-h, h)$ $(h > 0)$. We have

$$\sigma_N(0) = \frac{1}{2\pi} \int_{-h}^{h} f(y) K_N(y)\, dy + \frac{1}{2\pi} \int_{-\pi}^{-h} (\ ) + \frac{1}{2\pi} \int_{h}^{\pi} (\ ).$$

Observe that the first of the last three integrals is bounded by M, and that the other two converge to zero as $N \to \infty$ by the Lebesgue dominated convergence theorem. Therefore the $\sigma_N(0)$ are uniformly bounded. From this point on we proceed exactly as in the proof of Theorem 2, by letting $N \to \infty$ in (2). Thus $f \in \mathfrak{A}$. Now suppose $\alpha \neq 0$. Then f_α is essentially bounded in a neighborhood of the origin so that by the previous result, $f_\alpha \in \mathfrak{A}$. Since $\hat{f}(n) = e^{-in\alpha} \hat{f}_\alpha(n)$ we have $f \in \mathfrak{A}$. Next assume $f \in \mathfrak{A}$. Then f is essentially bounded on T and condition (**) is clearly satisfied.

Corollary 5. *Let* $f \in L^1(T)$. *Then* f *is equal almost everywhere to a linear combination of positive definite functions if and only if* (**) *is satisfied.* □

Remark.[1] Call a numerical series $\Sigma(a_n + ib_n)$ "one-sidedly absolutely

[1]The author wishes to thank the referee for suggesting this remark.

convergent'' if (at least one of Σa_n^+ or Σa_n^-) and (at least one of Σb_n^+ or Σb_n^-) is finite. The main theorems (2 and 4) are interesting because it is possible for $\Sigma e^{i\lambda} c_n$ to be one-sidedly absolutely convergent even when Σc_n is not. For example, $c_{2n} = 1 + i$, $c_{2n+1} = 1 - i$, $n = 0, 1, 2, \cdots$, and $\lambda = \pi/4$.

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