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RUDIN VARIETIES IN PRODUCTS OF TWO ANNULI[1]

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ABSTRACT. Let Q be an annulus and ∂Q its boundary. If f is holomorphic in $Q \times Q$ and its zero set is bounded away from $\partial Q \times \partial Q$, then there exists a bounded holomorphic function F with the same zeros as f such that F^{-1} is bounded near $\partial Q \times \partial Q$.

I. Introduction. Let Ω be a domain in $\mathbf{C}^n$ and $\check{\partial}\Omega$ be the Šilov boundary of the algebra of holomorphic functions in Ω with continuous boundary values. A subset V of Ω is a Rudin variety if it is the zero set of a holomorphic function in Ω, and if none of its limit points lie in $\check{\partial}\Omega$.

It is known [1], [3], that in a polydisc a Rudin variety is necessarily the zero set of a bounded holomorphic function. We show here that this is also the case in a product of two annuli. Unfortunately the method used does not extend to products of more than two annuli.

We use the notation of [1]. The complex numbers will be denoted by $\mathbf{C}$. Let $0 < r_1 < r_2 < \infty$ and define

$$Q(r_1, r_2) = \{z \in \mathbf{C} : r_1 < |z| < r_2\},$$

$$\partial Q(r_1, r_2) = \{z \in \mathbf{C} : |z| = r_1 \text{ or } |z| = r_2\}.$$

For any two sets $S_1 \subset \mathbf{C}$ and $S_2 \subset \mathbf{C}$, $S_1 \times S_2$ denotes their cartesian product. Points of $\mathbf{C} \times \mathbf{C}$ will be denoted by (z_1, z_2) or simply by z. Whenever Ω is a domain in $\mathbf{C} \times \mathbf{C}$, $H(\Omega)$, $H^\infty(\Omega)$, and $h^\infty(\Omega)$ denote the classes of holomorphic, bounded holomorphic, and of holomorphic functions with bounded real parts, respectively. If $f \in H(\Omega)$ its zero set $Z(f)$ is the set $\{z \in \Omega : f(z) = 0\}$. Two functions f_1, f_2 in $H(\Omega)$ are said to have the same zeros if their quotient $f_1 f_2^{-1}$ is an invertible holomorphic function in Ω. For any function f, $\operatorname{Re} f$ denotes its real part. Finally, the exponential function will be denoted by exp.

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II. Two lemmas. We start with two lemmas. For the first we state the result and refer the reader to [1, p. 92] for a proof.

Fix $0 < r_1 < r_2 < \infty$, let $U_1 = \{\lambda \in \mathbf{C} : |\lambda| > r_1\}$ and $U_2 = \{\lambda \in \mathbf{C} : |\lambda| < r_2\}$. Let Q be an arbitrary annulus centered at the origin.

Lemma 1. *Any $g \in h^\infty((U_1 \cap U_2) \times Q)$ can be written $g = g_1 - g_2$ where $g_1 \in h^\infty(U_1 \times Q)$ and $g_2 \in h^\infty(U_2 \times Q)$.*

The next lemma proves that a certain type of multiplicative Cousin problem with bounded data can always be solved in a product of two annuli (the solution being a bounded holomorphic function). Without loss of generality the two annuli will be centered at the origin and have the same inner and outer radii, respectively.

Lemma 2. *Let Q be an annulus centered at the origin and $\{A_\alpha\}$ a finite covering of $Q \times Q$ consisting of products of annuli centered at the origin. If on each A_α we are given a function $f_\alpha \in H^\infty(A_\alpha)$, and for all α and β, $f_\alpha f_\beta^{-1}$ is invertible in $H^\infty(A_\alpha \cap A_\beta)$, there exists a function $F \in H^\infty(Q \times Q)$ such that for every α, Ff_α^{-1} is invertible in $H^\infty(A_\alpha)$.*

Proof. Let $Q = Q(r, r + \epsilon)$ and define

$$Q_1 = Q(r, r + 2\epsilon/3), \qquad Q_2 = Q(r + \epsilon/3, r + \epsilon).$$

Suppose the induced "Cousin problem" can be solved in both $Q_1 \times Q$ and $Q_2 \times Q$ with solutions F_1 and F_2, respectively. Since for each α, $F_1 f_\alpha^{-1}$ and $f_\alpha F_2^{-1}$ are invertible in $H^\infty(A_\alpha \cap ((Q_1 \cap Q_2) \times Q))$, $F_1 F_2^{-1}$ will be invertible in $H^\infty((Q_1 \cap Q_2) \times Q)$.

Fix $r + \epsilon/3 < \rho < r + 2\epsilon/3$, and for each $t \in [0, 1]$ define $\Gamma_1(t) = (\rho e^{2\pi i t}, \rho)$ and $\Gamma_2(t) = (\rho, \rho e^{2\pi i t})$. Clearly Γ_1 and Γ_2 constitute a basis for the fundamental group of $(Q_1 \cap Q_2) \times Q$. Let $-\alpha_1$ and $-\alpha_2$ be the indices of $(F_1 F_2^{-1}) \circ \Gamma_1$ and $(F_1 F_2^{-1}) \circ \Gamma_2$ (both of which are loops in $\mathbf{C} - \{0\}$), and define

$$G(z) = F_1(z) F_2(z)^{-1} z_1^{\alpha_1} z_2^{\alpha_2} \qquad (z \in (Q_1 \cap Q_2) \times Q). \tag{1}$$

It follows that $G \circ \Gamma$ has index 0 for every loop Γ in $(Q_1 \cap Q_2) \times Q$, therefore there exists $g \in H((Q_1 \cap Q_2) \times Q)$ such that

$$G(z) = \exp g(z) \qquad (z \in (Q_1 \cap Q_2) \times Q).$$

Since G is invertible in $H^\infty((Q_1 \cap Q_2) \times Q)$, $\operatorname{Re} g$ must be bounded. Let $U_1 = \{\lambda \in \mathbf{C} : |\lambda| > r + \epsilon/3\}$ and $U_2 = \{\lambda \in \mathbf{C}: |\lambda| < r + 2\epsilon/3\}$; g is in

$h^\infty((Q_1 \cap Q_2) \times Q)$ and $U_1 \cap U_2 = Q_1 \cap Q_2$, so by Lemma 1 $g = g_1 - g_2$ with $g_1 \in h^\infty(U_1 \times Q)$ and $g_2 \in h^\infty(U_2 \times Q)$. Hence (1) can be written as

$$\exp(g_1(z) - g_2(z)) = F_1(z) F_2(z)^{-1} z_1^{\alpha_1} z_2^{\alpha_2} \qquad (z \in (Q_1 \cap Q_2) \times Q)$$

which enables us to define a function $F \in H^\infty(Q \times Q)$ by

$$F(z) = \begin{cases} F_1(z) \exp(-g_1(z)) z_1^{\alpha_1} z_2^{\alpha_2} & \text{if } z \in Q_2 \times Q, \\ F_2(z) \exp(-g_2(z)) & \text{if } z \in Q_1 \times Q. \end{cases}$$

It is easy to see that for each α, $F f_\alpha^{-1}$ is invertible in $H^\infty(A_\alpha)$, so if the induced "Cousin problem" can be solved in both $Q_1 \times Q$ and $Q_2 \times Q$ it can also be solved in $Q \times Q$.

Suppose now that our lemma is false. By what we just mentioned the induced "Cousin problem" cannot be solved in both $Q_1 \times Q$ and $Q_2 \times Q$; suppose it cannot be solved in $Q_1 \times Q$. Let $Q_1^1 = Q_1 \times Q$, $Q_2^1 = Q_1 \times Q_1$, $Q_2^2 = Q_1 \times Q_2$. Arguing as before, the induced "Cousin problem" would not be solvable in both Q_2^1 and Q_2^2, so on one of these, call it $Q_2^{k_2}$, it will be unsolvable. Iterating this procedure, proceeding cyclicly through the complex coordinates z_1 and z_2, we obtain a nested sequence

$$Q_1^1 \supset Q_2^{k_2} \supset Q_3^{k_3} \supset \cdots \supset Q_m^{k_m} \supset \cdots$$

of products of annuli centered at the origin whose thinness eventually decreases to zero, on none of which we are able to solve the induced problem. Since $\{A_\alpha\}$ is a finite collection of products of annuli centered at the origin and $Q \times Q = \bigcup A_\alpha$, for some integer m and some α we will have $Q_m^{k_m} \subset A_\alpha$. But f_α solves the induced "Cousin problem" in A_α (and hence in $Q_m^{k_m}$) so we have a contradiction. Consequently the lemma is true.

III. Rudin varieties in a product of two annuli. Fix $0 < r_1 < r_2 < \infty$ and let $Q = Q(r_1, r_2)$, $\partial Q = \partial Q(r_1, r_2)$.

Theorem 1. *Let $f \in H(Q \times Q)$. If $Z(f)$ has no limit points in $\partial Q \times \partial Q$ then there exists $F \in H^\infty(Q \times Q)$ with the same zeros as F, moreover F^{-1} is bounded near $\partial Q \times \partial Q$.*

Proof. Fix $0 < \epsilon < (r_2 - r_1)/2$ sufficiently small so that

$$(Q(r_1, r_1 + \epsilon) \cup Q(r_2 - \epsilon, r_2)) \times (Q(r_1, r_1 + \epsilon) \cup Q(r_2 - \epsilon, r_2))$$

does not intersect $Z(f)$, and define

$$Q_1 = Q(r_1, r_1 + \epsilon/2), \qquad Q_2 = Q(r_2 - \epsilon/2, r_2).$$

For each $z_1 \in Q_1(r_1, r_2 + \epsilon)$ the function $z_2 \to f(z_1, z_2)$ has finitely many zeros, all in $r_1 + \epsilon < |z_1| < r_2 - \epsilon$. Let D_2 denote differentiation with respect to z_2 and for each $k = 0, 1, 2, \cdots$ define

$$S_k(z_1) = \frac{1}{2\pi i}\left(\int_{\gamma_1} + \int_{\gamma_2}\right)\frac{D_2 f(z_1, \xi)}{f(z_1, \xi)}\, \xi^k \, d\xi$$

where

$$\gamma_1(t) = (r_1 + \epsilon/2)e^{-2\pi i t}, \qquad \gamma_2(t) = (r_2 - \epsilon/2)e^{2\pi i t} \qquad (t \in [0, 1]).$$

Note that each $S_k(z_1)$ is holomorphic in Q_1, and that by the residue theorem if $\alpha_1(z_1), \alpha_2(z_1), \cdots, \alpha_p(z_1)$ are the zeros of $z_2 \to f(z_1, z_2)$ then $S_k(z_1) = \Sigma_{i=1}^p \alpha_i(z_1)^k$. Since $S_0(z_1)$ is the number of zeros of $z_2 \to f(z_1, z_2)$ (counted with their multiplicities), it is an integer valued holomorphic and therefore constant function, i.e. $S_0(z_1) = p$ for all $z_1 \in Q_1$.

For each $z_1 \in Q_1$ and all z_2 define

$$\phi_{11}(z) = \prod_{i=1}^{p} (z_2 - \alpha_i(z_1)) = z_2^p + b_1(z_1)z_2^{p-1} + \cdots + b_p(z_1).$$

The connection between the functions b_j and S_j is given by Newton's identities

$$b_j(z_1) = -\frac{-1}{j}\,(S_j(z_1) + S_{j-1}(z_1)b_1(z_1) + \cdots + S_1(z_1)b_{j-1}(z_1))$$

for $1 \le j \le p$ (see for instance [1, pp. 11, 12]), which show that the coefficients b_j are holomorphic in Q_1, and hence that ϕ_{11} is holomorphic in $Q_1 \times \mathbf{C}$. Also

$$(\epsilon/2)^p < |\phi_{11}(z)| < (2r_2)^p \quad \text{if } z \in Q_1 \times (Q_1 \cup Q_2).$$

If we define $h(z) = F(z)/\phi(z)$ then $h(z)$ has no zeros and is holomorphic in z_2; thus

$$h(z) = \frac{1}{2\pi i}\left(\int_{\gamma^1} + \int_{\gamma^2}\right)\frac{F(z_1, \xi)}{\phi_{11}(z_1, \xi)}\,\frac{d\xi}{\xi - z_2} \qquad \left(r_1 + \frac{\epsilon}{2} < |z_2| < r_2 - \frac{\epsilon}{2}\right),$$

which shows that $h(z)$ is holomorphic in z_1, and hence holomorphic in $Q_1 \times Q$. We have $\phi_{11} \in H^\infty(Q_1 \times Q)$, and similarly define $\phi_{12} \in H^\infty(Q_2 \times Q)$, $\phi_{21} \in H^\infty(Q \times Q_1)$, $\phi_{22} \in H^\infty(Q \times Q_2)$. All of them recapture the zeros of f in their respective domains, and the quotient of any two, wherever it can be considered, is an invertible bounded holomorphic function.

Define now $Q_3 = Q(r_1 + \epsilon/3, r_2 - \epsilon/3)$ and denote the restriction of f to $Q_3 \times Q_3$ by Ψ. Then, by what we have said above, $\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}$ and Ψ, and their domains, satisfy the same properties as the functions f_α and the sets V_α of Lemma 3. Thus there exists a function $F \in H^\infty(Q \times Q)$ such that $\phi_{11}F^{-1}, \phi_{12}F^{-1}, \phi_{21}F^{-1}, \phi_{22}F^{-1}$ and ΨF^{-1} are invertible bounded holomorphic functions. Therefore F has the same zeros as f, and since ϕ_{11} is bounded away from zero in $Q_1 \times (Q_1 \cup Q_2)$, F^{-1} is bounded there. Similarly F^{-1} is bounded in $Q_2 \times (Q_1 \cup Q_2)$. The theorem then is proved since we have shown the existence of $F \in H^\infty(Q \times Q)$ with the same zero as f, and that F^{-1} is bounded in $(Q_1 \cup Q_2) \times (Q_1 \cup Q_2)$.

IV. **Comments.** From Theorem 1 it follows that in $Q \times Q$ a Rudin variety is the zero set of a bounded holomorphic function. For higher complex dimension Lemma 3 still holds, but the Weierstrass polynomials (which can be constructed exactly the same as in Theorem 1), together with f restricted to a "smaller" polyannulus, do not constitute appropriate data for Lemma 3, in the sense that their domains do not cover all of the original polyannulus.

For a polydisc the analogue to Theorem 1 (proved for arbitrary complex dimension) is due to W. Rudin [1], [3]. For complex dimension two a similar scheme to the one above can be used to prove this result, the solvability of the second Cousin problem with bounded data for a polydisc (proved by E. Stout in [4]) playing the role of Lemma 3. This is easy to see, but as with a product of annuli, this method does not appear to be applicable in higher complex dimension.

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