ON SOME RINGS WHOSE MODULES
HAVE MAXIMAL SUBMODULES

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ABSTRACT. It is shown that a principal right ideal domain, having
the property that every right R module has a maximal submodule must
be simple. Strong conditions satisfied by these rings are deduced giving
evidence for the conjecture that they must be V-rings. We also general-
ize an example of Faith by showing that a subring of an infinite dimen-
sional full linear ring, which contains the socle of that ring is never a
left V-ring.

Cozzens [1] and Koĭfmann [3] gave striking examples of simple prin-
cipal right ideal domains which are right *V-rings* (every simple module is
injective). These examples answer two questions of Faith, whether there
exists a nonsemisimple, simple, noetherian V-ring, and whether every
V-ring is regular; and Bass' question, whether a ring R over which every
right R module has a maximal submodule (*a max ring*) must be right per-
fect. Interest in these examples is enhanced by the fact that, being simple
domains, they are highly nonregular and highly nonperfect, thus answering
the last two questions, very reasonable ones given the state of knowledge
at the time, very definitively in the negative. They show that in the non-
commutative case (see [3] for the commutative case) the hypothesis "R
is a max ring" does not seem to place very strong conditions on R.

Now the most general proof of the fact that these rings are simple
comes from a theorem of Faith [2, p. 130], that any V-ring which is an
order in a semisimple ring must be a product of simple rings. Our first
purpose is to show that in fact if R is a right P.I.D., as the examples
cited are, then one needs only assume that R is a right max ring in order
to conclude that R is simple. This result seems interesting because it
seems to be an exception to the apparent weakness of the max ring hypoth-
esis as cited in the previous paragraph.

The question that naturally arises from this is: Are these rings simple because they are V-rings? This problem seems to be quite difficult. We are able, however, to present some strong evidence for its truth. We show that if R is a right P.I.D. and a right max ring then for all $a, b \in R$ which generate maximal right ideals, either R/abR or R/baR is semisimple. This is interesting because, as is explained in the text (before Proposition 2), if R is also a left P.I.D. then R is a V-ring if and only if R/abR is semisimple for all such a and b.

The second section is motivated by the example given by Faith of a regular ring which is not a V-ring. (A commutative ring is a V-ring if and only if it is regular.) Faith's example is the subring of any full linear ring $L = \text{End } V_F$, F a field, generated by the socle of L and the scalars. We generalize this by showing directly that if $L = \text{End } V_D$, D a division ring, and if R is any subring of L containing socle L, then R is not a left V-ring.

I. Theorem 1. *If R is a right max ring, which is a right P.I.D., then R is simple.*

Proof. Let dR be a two-sided ideal in R, and let Q be the classical right quotient ring of R. Let $U \subset Q$ be given by $U = \bigcup_{n \in N} d^{-n}R$. Then U/R has a maximal submodule M/R. Now, there is an n with $d^{-n+1} \in M$, but $d^{-n} \notin M$. However, since U/M is simple, $d^{-n-1} + M = d^{-n}r + M$ for some $r \in R$. But $rd = ds$ for some s, so that $d^{-1}r = sd^{-1}$. Thus, $d^{-n-1} + M = d^{-n+1}sd^{-1} + M$ and multiplying on the right by d we have $d^{-n} + M = d^{-n+1}s + M$.

However, the right side is zero since $d^{-n+1} \in M$, so $d^{-n} \in M$. This shows that $U/R = 0$, or that $d^{-1} \in R$, so that $dR = R$.

The previous result suggests the following: If R is a P.I.D., is R a V-ring if and only if it is a max ring? This question seems difficult. Below, however, we present a proposition which is quite suggestive. The reason for this is that if R is a two-sided P.I.D. one can show in a relatively straightforward manner that R is a right V-ring if and only if R/abR is semisimple for all a and b which generate maximal right ideals. (Just determine that the injective hull of a simple module must have a submodule of length two, which must be of the form R/abR.)

Proposition 2. *Let R be a right max ring which is also a right Ore domain with classical right quotient ring Q. Then:*

(a) *If $a, b \in R$ and a and b generate maximal right ideals, either R/abR or R/baR is semisimple.*

(b) *If aR is a maximal right ideal then $R/a^n R$ is semisimple for all $n \in N$.*

Proof. (a) Use the same idea as in the proof of Theorem 1. Consider the chain $a^{-1}R \subset a^{-1}b^{-1}R \subset a^{-1}b^{-1}a^{-1}R \cdots$ in Q, and let U be the union. Let M/R be a maximal submodule of U/R. Then there is an n such that either:

(1) $(a^{-1}b^{-1})^{n-1} \in M$ but $(a^{-1}b^{-1})^{n-1}a^{-1} \notin M$, or

(2) $a^{-1}(b^{-1}a^{-1})^{n-1} \in M$ but $a^{-1}(b^{-1}a^{-1})^{n-1}b^{-1} = (a^{-1}b^{-1})^n \notin M$.

If (1) occurs, let $x = (a^{-1}b^{-1})^n$. Then $(x : M) = \{r \in R \,|\, xr \in M\}$ is a maximal right ideal in R which contains ba. It does not contain b, or else $(a^{-1}b^{-1})^{n-1}a^{-1} \in M$. Thus

$$R/baR = bR/baR \oplus (x : M)/baR.$$

If case (2) occurs, an analogous argument shows that R/abR is semisimple.

(b) Again let $U = \bigcup_{a \in N} a^{-n}R$ and let M/R be a maximal submodule of U/R. Then there is a k such that $a^{-k} \notin M$, but $a^{-k+1} \in M$. Look at $C = (a^{-m-k} : M)$. Then C is a maximal right ideal containing a^{n+1} but not a^n. Since $a^n R/a^{n+1}R \quad R/aR$ is simple we have

$$R/a^{n+1}R = a^n R/a^{n+1}R \oplus C/a^{n+1}R.$$

Then, $C/a^{n+1}R \approx R/a^n R$, which we may assume to be semisimple by induction, so that $R/a^{n+1}R$ is.

Remarks. (A) Proposition 2 may be proved under the hypothesis that R is a semiprime principal right ideal ring.

(B) Theorem 1 also works for a prime Goldie ring which is right principal since we only require that the ideal in question be generated by a regular element.

(C) If R is a principal right ideal domain with a unique simple module then, in fact, R is a right max ring if and only if R is a right V-ring, for the right quotient ring Q of R is injective and has a maximal submodule M. Then Q/M is injective since R is hereditary. The idea for this is found in [3, Proposition 1.1].

II. Writing functions opposite scalars we have the following generalization of Faith's example, as explained in the introduction. Recall the socle of a full linear ring is a collection of all linear transformations of finite rank, a fact we use without explicit mention below.

Theorem 3. *If R is a subring of an infinite dimensional full linear ring $L = \text{End } W_D$, containing socle L, then R is not a left V-ring.*

Proof. Let X be a basis and pick out countable subset $\{x_i | x_i \in N\}$. For each $i \in N$ let $f_i \in R$ be defined by $f_i(x_i) = x_1$ and $f_i(x) = 0$ for $x \in X$; $x \neq x_i$. Then $\Sigma R f_i$ is direct, for if $p_1 f_1 + \cdots + p_n f_n = 0$ then, for any k we have

$$p_1 f_1(x_k) + \cdots + p_k f_k(x_k) + \cdots + p_n f_n(x_k) = 0$$

and, since $p_i f_i(x_k) = 0$ for $i \neq k$, we have $p_k f_k(x_k) = 0$ and thus, $p_k f_k(x) = 0$ for all $x \in X$ so that $p_k f_k = 0$.

Further, if $l(g)$ denotes the left annihilator of $g \in L$, then $l(f_k) \subset l(f_1)$. In fact, if $\text{Im } L_1 \supset \text{Im } L_2$ for any two linear transformations then $l(L_1) \subset l(L_2)$, for if $pL_1 = 0$, then for any $x \in W$, $L_2(x) = L_1(y)$ so that $pL_2(x) = pL_1(y) = 0$ and $pL_2 = 0$.

Thus, the maps H_i defined by $(pf_i)H_i = pf_1$ are well defined. Since $\Sigma R f_i$ is direct, this yields a map $H: \Sigma R f_i \to R f_1$ given by

$$\left(\sum p_i f_i\right)H = \sum (p_i f_i)H_i = \left(\sum p_i\right)f_1.$$

Now, Rf_1 is simple. For if $pf_1 \neq 0$ then $pf_1(x_1) \neq 0$ so there is a linear transformation g with $gpf_1(x_1) = x_1$ and $\text{Im } g \subset x_1 D$. Then $g \in R$ since R contains socle L. But clearly, from the construction of f_1, $gpf_1 = f_1$ so that $Rpf_1 = Rf_1$ and thus Rf_1 is simple.

We now show that the map H cannot be extended and that Rf_1 is therefor a simple noninjective module. For, let (1) $H = tf_1$ and let $tf_1(x_1) = x_1 d_1 + \cdots + x_n d_n$. Then, $f_1 = (f_{n+1})H = f_{n+1}tf_1$ so that $f_1 = f_{n+1}tf_1$ or $f_1(x_1) = f_{n+1}tf_1(x_1)$ or

$$x_1 = f_{n+1}\left(\sum_{i=1}^n x_i d_i\right) = \sum_{i=1}^n f_{n+1}(x_i)d_i = 0,$$

a contradiction.

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