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A RANK THEOREM FOR INFINITE DIMENSIONAL SPACES

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ABSTRACT. Suppose X is a Banach space, U is an open set of X containing 0, and f is a continuously differentiable function from U into X satisfying $f(0) = 0$ and $f'(0)^2 = f'(0)$. An additional hypothesis is given for f which, in case X is finite dimensional, is equivalent to assuming $\operatorname{rank} f'(x) = \operatorname{rank} f'(0)$ for all x in some neighborhood of 0. Under this hypothesis one obtains a local factorization of f into $h_1 \circ f'(0) \circ h_2$ where each of h_1 and h_2 is a continuously differentiable homeomorphism. In addition there is a neighborhood of 0 in $f^{-1}(\{0\})$ which is the image of a continuously differentiable retraction. An application of these results to the theory of differentiable multiplications is given.

Suppose X is a Banach space, U is an open set of X containing 0, and f is a continuously differentiable function from U into X satisfying $f(0) = 0$ and $f'(0)^2 = f'(0)$. An additional hypothesis is given for f which, in case X is finite dimensional, is equivalent to assuming $\operatorname{rank} f'(x) = \operatorname{rank} f'(0)$ for all x in some neighborhood of 0. Under this hypothesis one obtains a local factorization of f into $h_1 \circ f'(0) \circ h_2$ where each of h_1 and h_2 is a continuously differentiable homeomorphism. In addition there is a neighborhood of 0 in $f^{-1}(\{0\})$ which is the image of a continuously differentiable retraction. An application of these results to the theory of differentiable multiplications is given.

Condition I. There is a neighborhood W of 0 in X so that $(f'(0)|f(W))$ is one to one.

Condition II. There is a neighborhood W of 0 in X so that if V is a neighborhood of 0 in $f'(0)(X)$ then $f(V)$ is a neighborhood of 0 in $f(W)$.

Theorem 1. *f satisfies Condition* I *if and only if f satisfies Condition* II.

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Proof. Let N denote the image of $f'(0)$. N is closed since $f'(0)^2 = f'(0)$ and hence is a Banach space. $(f'(0) \circ f|N)'(0) = (f'(0) \circ f'(0)|N) = (f'(0)|N)$ is the identity function on N. Thus, by the inverse function theorem [1, p. 268], there are neighborhoods A and B of 0 in N so that $(f'(0) \circ f|A)$ is a homeomorphism onto B and $(f'(0) \circ f|A)^{-1}$ is continuously differentiable on B.

Now suppose f satisfies Condition I and W' is a neighborhood of 0 in X so that $(f'(0)|f(W'))$ is one to one. Choose a neighborhood W of 0 in X so that W is contained in W' and $W \cap N$ is contained in A.

Suppose V is a neighborhood of 0 in N and V is contained in W. $f'(0)(f(V))$ is a neighborhood of 0 in N since V is contained in A. Let $C = f'(0)^{-1}(f'(0)(f(V))) \cap f(W)$, and suppose y is in C. $f'(0)(y)$ is in B so there is an x in A so that $f'(0)(y) = f'(0)(f(x))$. But each of $f(x)$ and y is in $f(W)$ so $f(x) = y$ since $f(W)$ is contained in $f(W')$.

Suppose f satisfies Condition II and choose W' so that if V is contained in W' and V is a neighborhood of 0 in N then $f(V)$ is a neighborhood of 0 in $f(W')$. Since $f(A \cap W')$ is a neighborhood of 0 in $f(W')$ we may choose the neighborhood W of 0 in X so that $f(W)$ is contained in $f(A \cap W')$.

If each of x and y is in $f(W)$ there are members z and w of A so that $f(z) = x$ and $f(w) = y$. If $f'(0)(x) = f'(0)(y)$ then $f'(0)(f(z)) = f'(0)(f(w))$ and hence $z = w$ since each of z and w is in A. Thus $(f'(0)|f(W))$ is one to one.

Note. If A is chosen as before and C is a closed subset of A then $(f|C)$ is a homeomorphism onto $f(C)$. This is a consequence of the following observations. If $\{a_n\}$ is a sequence in C and $\{f(a_n)\}$ converges then $\{f'(0)f(a_n)\}$ converges to some member y of $f'(0)(f(C))$. Thus $\{a_n\}$ converges to $(f'(0) \circ f|A)^{-1}(y)$ and $(f|C)^{-1}$ is continuous.

Theorem 2. *If $f(f(x)) = f(x)$ for each x in $U \cap f^{-1}(U)$ then f satisfies Condition* II.

This is a consequence of Lemma 3 in [2].

Theorem 3. *If f satisfies Condition* I *then there is a neighborhood C of 0 in X and continuously differentiable homeomorphisms h_1 and h_2, each from a neighborhood of 0 in X onto a neighborhood of 0 in X, so that $(f|C) = (h_1 \circ f'(0) \circ h_2|C)$.*

Proof. Since f satisfies Conditions I and II there is a neighborhood W of 0 in X so that if V is a neighborhood of 0 in N and V is contained in W then $f(V)$ is a neighborhood of 0 in $f(W)$ and $(f'(0)|f(W))$ is one to one.

Choose neighborhoods A and B of 0 in N so that A is contained in W, $(f'(0) \circ f|A)$ is a homeomorphism onto B, and $(f'(0) \circ f|A)^{-1}$ is continuously differentiable on B.

Define K_2 on W by $K_2(x) = f(x) + (I - f'(0))(x)$. ($I$ denotes the identity function on X.) $K_2'(0) = I$ so by the inverse function theorem, there are neighborhoods D and E of 0 in X so that $(K_2|D) = h_2$ is a homeomorphism onto E and $f'(0)(E)$ is contained in B.

Define K_1 by

$$K_1(x) = (f'(0)|f(A))^{-1}(f'(0)(x)) + (I - f'(0))(x)$$

for each x in $f'(0)^{-1}(A \cap B)$. K_1 is well defined since $f(A)$ is contained in $f(W)$. If x is in $A \cap B$ then

$$(f|A) \circ (f'(0) \circ f|A)^{-1}(x) = (f'(0)|f(A))^{-1}(x).$$

Thus $(f'(0)|f(A))^{-1} \circ f'(0)$ is continuously differentiable on $\mathrm{dom}(K_1)$.

In particular, $K_1'(0) = I$, so by the inverse function theorem there are neighborhoods F and G of 0 in X so that $h_1 = (K_1|F)$ is a homeomorphism onto G.

Choose C, a neighborhood of 0 in X, so that C is contained in D, $f'(0)(h_2(C))$ is contained in $\mathrm{dom}(h_1)$, and $f(C)$ is contained in $f(A)$. (This last condition can be arranged since $f(A)$ is a neighborhood of 0 in $f(W)$.)

If x is in C then

$$h_1(f'(0)(h_2(x))) = (f'(0)|f(A))^{-1}(f'(0)(f(x))) = f(x).$$

Compare this conclusion with that of the rank theorem [1, p. 277].

Suppose X is finite dimensional, V is an open set of X, x_0 is in V, and g is a continuously differentiable function from V to X satisfying $\mathrm{rank}(g'(x)) = \mathrm{rank}(g'(x_0))$ for each x in V. There is a linear homeomorphism T from X into X so that the function f defined by $f(x) = T(g(x + x_0) - g(x_0))$ for each x in $U = V - x_0$ satisfies $f'(0)^2 = f'(0)$, $f(0) = 0$, and $\mathrm{rank}(f'(x)) = \mathrm{rank}(f'(0))$ for each x in U. Thus, the following theorem may be applied to g.

Theorem 4. *If X is finite dimensional then f satisfies Condition* I *if and only if* $\mathrm{rank}(f'(x)) = \mathrm{rank}(f'(0))$ *for each x in some neighborhood of* 0.

Proof. Suppose X is finite dimensional and f satisfies Condition I. Choose homeomorphisms h_1 and h_2 as in the conclusion of Theorem 3. Since $h_1'(0) = h_2'(0) = I$ there is a neighborhood D of 0 in X so that if x is in D then each of $h_1'(f'(0)(h_2(x)))$ and $h_2'(x)$ is invertible. If x is in D then $f'(x) = h_1'(f'(0)(h_2(x))) \circ f'(0) \circ h_2'(x)$, and hence $\mathrm{rank}(f'(x)) = \mathrm{rank}(f'(0))$.

Now suppose X is finite dimensional, D is a neighborhood of 0 in X, and $\operatorname{rank}(f'(x)) = \operatorname{rank}(f'(0))$ for each x in D. Let $N = f'(0)(X)$ and $M = \ker(f'(0)) = (I - f'(0))(X)$. Define h from D into $N \times M$ by $h(x) = (f'(0)(f(x)), (I - f'(0))(x))$. Regard $N \times M$ as a Banach space in the usual way and note that $h'(0) = f'(0) \times (I - f'(0))$ is a linear homeomorphism onto $N \times M$. By the inverse function theorem, we may choose E a neighborhood of 0 in X so that $(h|E)$ is a homeomorphism onto a neighborhood $(0, 0)$ in $N \times M$, $(h|E)^{-1}$ is continuously differentiable on $h(E)$, and $[(h|E)^{-1}]'(x) = [h'((h|E)^{-1}(x))]^{-1}$ for each x in $h(E)$.

If x is in E then $h'(x)(n) = (f'(0) \circ f'(x)(n), 0)$ for each n in N. $h'(x)$ is invertible so $\dim(h'(x)(N)) = \dim(N) = \dim(N \times \{0\})$. Thus $f'(0) \circ f'(x)(N) = N$. This implies that each of $(f'(0)|f'(x)(X))$ and $(f'(x)|N)$ is one to one onto its image since $\operatorname{rank}(f'(x)) = \operatorname{rank}(f'(0))$.

If x is in E then $f(x) = f(h^{-1}(h(x)))$. Hence, by the chain rule,

$$f'(x) = (f \circ h^{-1})'(h(x)) \circ h'(x)$$

$$= (D_1 f \circ h^{-1})(h(x)) \circ f'(0) \circ f'(x) + (D_2 f \circ h^{-1})(h(x)) \circ (I - f'(0)).$$

So

$$(D_2 f \circ h^{-1})(h(x)) \circ (I - f'(0)) = f'(x) - (D_1 f \circ h^{-1})(h(x)) \circ f'(0) \circ f'(x)$$

$$= [(f'(0)|f'(x)(X))^{-1} - (D_1 f \circ h^{-1})(h(x))] f'(0) \circ f'(x).$$

If n is in N then there is a y in N so that $f'(0)(f(x)(y)) = n$. Hence

$$[(f'(0)|\operatorname{im}(f'(x)))^{-1} - (D_1 f \circ h^{-1})(h(x))](n)$$

$$= (D_2 f \circ h^{-1})(h(x)) \circ (I - f'(0))(y) = 0.$$

Thus $(D_2 f \circ h^{-1})(h(x)) \circ (I - f'(0))(y) = 0$ for each y in X and $(D_2 f \circ h^{-1})(h(x)) = 0$ for each x in E.

Choose neighborhoods F and G of 0 in N and M, respectively, so that G is convex and $F \times G$ is contained in $h(E)$. If each of (x, y) and (x, z) is in $F \times G$ then

$$f(h^{-1}(x, y)) - f(h^{-1}(x, z)) = \int_0^1 dt[(f' \circ h^{-1})'(x, z + t(y - z))(0, y - z)]$$

$$= \int_0^1 dt[(D_1 f \circ h^{-1})(x, z + t(y - z))(0)$$

$$+ (D_2 f \circ h^{-1})(x, z + t(y - z))(y - z)] = 0$$

since $(x, z + t(y - z))$ is in $h(E)$ for each t in $[0, 1]$.

Choose a neighborhood W of 0 in X so that $h(W)$ is contained in $F \times G$. If each of x and y is in W and $f'(0)(f(x)) = f'(0)(f(y))$ then

$$f(x) = f(h^{-1}(h(x))) = f(h^{-1}(h(y))) = f(y)$$

so $(f'(0)|f(W))$ is one to one.

The idea for the last part of this proof is contained in Dieudonné's proof of the rank theorem [1, p. 277].

Theorem 5. *Suppose f satisfies Condition* I. *There is a neighborhood S of 0 in X and a continuously differentiable function q with domain S so that $q(S)$ is a neighborhood of 0 in $f^{-1}(\{0\})$, $q \circ q = q$, and $(f \times q|S)$ is a homeomorphism onto a neighborhood of $(0, 0)$ in $f(S) \times f^{-1}(\{0\})$.*

Proof. By Conditions I and II we may choose a neighborhood W of 0 in X so that $(f'(0)|f(W))$ is one to one and if V is a neighborhood of 0 in N contained in W then $f(V)$ is a neighborhood of 0 in $f(W)$. Choose neighborhoods A and B of 0 in N so that $(f'(0) \circ f|A)$ is a homeomorphism onto B, $(f'(0) \circ f|A)^{-1}$ is continuously differentiable on B, and A is contained in W. Choose C and D neighborhoods of 0 in X so that each of C and D is contained in W, $f(C)$ is contained in $f(A)$, $(K_1|C)$ is a homeomorphism onto D, and $(K_1|C)^{-1}$ is continuously differentiable on D.

$f(A)$ is a neighborhood of 0 in $f(W)$ so $f(A) \times (I - f'(0))(W)$ is a neighborhood of $(0, 0)$ in $f(W) \times (I - f'(0))(W)$. Suppose each of (a, b) and (c, d) is in $f(A) \times (I - f'(0))(W)$ and $a + b = c + d$. There are x and y in A so that $f(x) = a$ and $f(y) = c$. $d - b$ is in $\mathrm{im}(I - f'(0)) = \ker(f'(0))$ so $f'(0)(f(x)) = f'(0)(f(y))$. Thus $x = y$, $a = c$, and $b = d$ so $(+|f(A) \times (I - f'(0))(W))$ is one to one.

Let g be defined on C by $g(x) = (f(x), (I - f'(0))(x))$. Then $(K_1|C) = (+ \circ g|C)$. Hence, $(g|C)$ is one to one and $+(g(C)) = D$.

$$E = +^{-1}(D) \cap [f(A) \times (I - f'(0))(W)]$$

is a neighborhood of $(0, 0)$ in $f(W) \times (I - f'(0))(W)$. Suppose e is in E. $+(e) = K_1(z)$ for some z in C so $+(e) = +(g(z))$. But each of e and $g(z)$ is in $f(A) \times (I - f'(0))(W)$ so $g(z) = e$. Thus $g(C)$ contains E.

Choose F and G neighborhoods of 0 in $f(W)$ and $(I - f'(0))(W)$, respectively, so that $F \times G$ is contained in $g(C)$. Let $h = (K_1^{-1}|G)$, and define q on $(I - f'(0))^{-1}(G)$ by

$$q(x) = h((I - f'(0))(x)) = g^{-1}(0, x - f'(0)(x)).$$

If x is in $(I - f'(0))^{-1}(G)$ then

$$g(q(x)) = g(g^{-1}(0, x - f'(0)(x))) = (0, x - f'(0)(x)).$$

Thus $q(x)$ is in $f^{-1}(\{0\}) \cap (I - f'(0))^{-1}(G)$.

$$q(q(x)) = g^{-1}(0, q(x) - f'(0)(q(x))) = g^{-1}(g(q(x))) = q(x)$$

if x is in $(I - f'(0))^{-1}(G)$ so $q \circ q = q$.

Suppose x is in $g^{-1}(F \times G) \cap f^{-1}(\{0\})$.

$$x = g^{-1}(g(x)) = g^{-1}(0, x - f'(0)(x)) = q(x - f'(0)(x)).$$

Thus $\mathrm{im}(q)$ is a neighborhood of 0 in $f^{-1}(\{0\})$.

Since $h = (K_1^{-1}|G)$ h is continuously differentiable on G, q is continuously differentiable on $(I - f'(0))^{-1}(G)$. Moreover, $q'(0) = h'(0) \circ (I - f'(0))$,

$$h'(0) = ([K_1'(K_1^{-1}(0))]^{-1}|\mathrm{im}(I - f'(0))) = (I - f'(0)|\mathrm{im}(I - f'(0)))$$

so $q'(0) = I - f'(0)$. From Lemma 3 of [2] there is a neighborhood H of 0 in $\mathrm{im}(q'(0))$ so that $(q|H)$ is a homeomorphism onto a neighborhood of 0 in $\mathrm{im}(q)$.

Choose S a neighborhood of 0 in X so that S is contained in $(I - f'(0))^{-1}(H \cap G)$ and $S \cap N$ is contained in A. Define r on $f(S) \times (I - f'(0))(S)$ by $r(x, y) = (x, q(y))$. By choice of H, r is a homeomorphism onto a neighborhood of 0 in $f(S) \times q(S)$. $(f \times q|S) = (r \circ g|S)$. Thus $(f \times q|S)$ is a homeomorphism onto a neighborhood of $(0, 0)$ in $f(S) \times q(S)$. This concludes the proof of Theorem 5.

We will now indicate an application of the preceding theorems to the local theory of differentiable semigroups. Suppose D is an open set of X containing 0, and V is a continuously differentiable associative function from $D \times D$ into X so that $V(0, 0) = 0$. Define f on D by $f(x) = V(x, 0)$. If each of x and $f(x)$ is in D then $f(f(x)) = f(x)$. Thus, by Theorem 2, f satisfies Condition I. By Theorem 5 then there is a neighborhood S of 0 in X and a continuously differentiable function q defined on S so that $q \circ q = q$ and $\mathrm{im}(q)$ is a neighborhood of 0 in $f^{-1}(\{0\})$. By Lemma 3 of [2] there is a neighborhood V of 0 in $Y = \mathrm{im}(q'(0)) = \mathrm{im}(I - f'(0))$ so that $(q|V)$ is a homeomorphism onto a neighborhood of 0 in $\mathrm{im}(q)$, $(q'(0) \circ q|V)$ is a homeomorphism onto a neighborhood of 0 in Y, and $(q'(0) \circ q|V)^{-1}$ is continuously differentiable. Define W contained in $(Y \times Y) \times Y$ by

$$W(x, y) = (q|V)^{-1}(V(q(x), q(y)))$$

whenever $V(q(x), q(y))$ is in $q(V)$. The domain of W is a neighborhood of $(0, 0)$ in $Y \times Y$, W is associative,

$$W(x, y) = (q|V)^{-1} \circ (q'(0)|q(V))^{-1} \circ q'(0) \circ V(q(x), q(y))$$

$$= (q'(0) \circ q|V)^{-1} \circ q'(0) \circ V(q(x), q(y))$$

so W is continuously differentiable on a neighborhood of $(0, 0)$, and q is a

local isomorphism between the local differentiable semigroups $(f^{-1}(\{0\})$, $(V|f^{-1}(\{0\}) \times f^{-1}(\{0\})))$ and (Y, W).

In [3] it was shown that $\mathrm{im}(f)$ is locally the topological and algebraic product of a local Lie group and a left trivial semigroup. Horne in [4] began a study of differentiable semigroups with right zero. The above shows that (X, V) is, near 0, the topological product of the differentiable subsemigroups $f(D)$ and $f^{-1}(\{0\})$.

Can V be reconstructed from its restriction to $(f(D) \times f(D)) \cup (f^{-1}(\{0\}) \times f^{-1}(\{0\}))$?

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