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COMPLETE INTERSECTION MANIFOLDS WITH EXTREMAL EULER-POINCARÉ CHARACTERISTICS

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ABSTRACT. Studies on n-dimensional complete intersection manifolds with Euler-Poincaré characteristic $n+1$.

1. **Introduction.** Let $P_{n+p}(C)$ be an $(n+p)$-dimensional complex projective space. An n-dimensional algebraic manifold imbedded in $P_{n+p}(C)$ is called a *complete intersection manifold* if M is given as an intersection of p nonsingular hypersurfaces $M_1, \cdots, M_p$ in general position in $P_{n+p}(C)$: $M = M_1 \cap \cdots \cap M_p$.

Let $b_i(M)$ be the ith Betti number of M. It is known that if M is an n-dimensional complete intersection manifold, then

$$b_{2k}(M) = 1 \qquad (2k \neq n),$$

$$b_{2k+1}(M) = 0 \qquad (2k+1 \neq n).$$

Therefore the Euler-Poincaré characteristic $\chi(M)$ of M satisfies $\chi(M) \geq n+1$ (resp. $\leq n+1$) provided that n is even (resp. odd). If M is a linear subspace or an odd-dimensional hyperquadric, then the equality holds.

The purpose of this paper is to consider the converse.

2. **Preliminaries.** Let $P_{n+p}(C)$ be an $(n+p)$-dimensional complex projective space and let M be an n-dimensional complete intersection manifold imbedded in $P_{n+p}(C)$: $M = M_1 \cap \cdots \cap M_p$, where M_α's are nonsingular hypersurfaces in $P_{n+p}(C)$. The following result gives a concrete formula for the Euler-Poincaré characteristic of complete intersection manifold.

Proposition 1 [1]. *Let $M = M_1 \cap \cdots \cap M_p$ be an n-dimensional complete intersection manifold imbedded in $P_{n+p}(C)$. If* $\deg M_\alpha = a_\alpha$, *then the Euler-Poincaré characteristic $\chi(M)$ of M is given by*

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$$\chi(M) = \left[\sum_{k=0}^{n} (-1)^k \binom{n+p+1}{n-k} \sigma_k\right]\left(\prod_{\alpha=1}^{p} a_\alpha\right),$$

where $\sigma_k = \sum_{\alpha_1 \leq \cdots \leq \alpha_k} a_{\alpha_1} \cdots a_{\alpha_k}$ *(the sum of all homogeneous monomials of degree* k *in* $a_1, a_2, \cdots, a_p$*) and* $\binom{n+p+1}{n-k}$ *is the binomial coefficient.*

The following is an immediate consequence of Proposition 1 (cf. [1, Theorem 3.1]).

Lemma 1. *Let* M *be an* n*-dimensional complete intersection manifold. If* $\chi(M) = \nu_1 \cdot \cdots \cdot \nu_p$ *for some prime numbers* $\nu_1, \cdots, \nu_p$ $(\neq \pm 1)$, *then* M *can be imbedded as a complete intersection manifold in* $P_{n+p}(C)$.

The smallest codimension for a complete intersection manifold is roughly estimated by the Euler-Poincaré characteristic.

Theorem 1. *Let* p *be the smallest integer for which an* n*-dimensional complex manifold* M *can be imbedded as a complete intersection manifold in* $P_{n+p}(C)$. *Then* $p \leq \log_2 |\chi(M)|$.

Proof. The assumption implies that $a_\alpha \geq 2$ for $\alpha = 1, \cdots, p$. Therefore it follows from Proposition 1 that $|\chi(M)| \geq 2^p$, i.e., $p \leq \log_2 |\chi(M)|$. Q.E.D.

The following is an answer to our problem in a special case.

Theorem 2. *Let* $M = M_1 \cap \cdots \cap M_p$ *be an* n*-dimensional complete intersection manifold imbedded in* $P_{n+p}(C)$. *If* n *is even,* $\chi(M) = n + 1$ *and* $\deg M_1 = \cdots = \deg M_p$, *then* M *is a linear subspace.*

Proof. Let $\deg M_1 = \cdots = \deg M_p = a$. Then $\sigma_k = \binom{p+k-1}{k} a^k$ so that

$$\chi(M) = \frac{(n+p+1)!}{n!(p-1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{a^{p+k}}{(p+k)(p+k+1)}.$$

Let $\omega(a)$ denote the function of a defined on the interval $[0, \infty)$ and given by the right-hand side of the above equation. We put

$$f(a) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{a^{p+k}}{(p+k+1)(p+k)}, \qquad g(a) = af(a),$$

and

$$h(a) = ag'(a) - g(a).$$

Then, by a straightforward computation, we have $h'(a) = a^p(1-a)^n$, from

which it follows that $h(a) > 0$, i.e., $ag'(a) - g(a) > 0$ for all $a > 0$. Therefore $f'(a) > 0$ for all $a > 0$ so that the function $\omega(a)$ as well as $f(a)$ is monotonically increasing. Since $\omega(1) = n + 1$, it follows that $\omega(a) > n + 1$ for $a > 1$. Thus $\chi(M) = n + 1$ only when $a = 1$. Q.E.D.

Immediately from Theorem 2, we have

Theorem 3. *Let p be the smallest integer for which an n-dimensional complex manifold M can be imbedded as a complete intersection manifold in $P_{n+p}(C)$. If n is even and $\chi(M) = n + 1$, then $p \leq 1 + \log_3[(n + 1)/5]$.*

3. **Low codimensional case.** The answer to the simplest case has been known as follows (cf. [3]).

Proposition 2. *Let M be a compact hypersurface imbedded in $P_{n+1}(C)$. If $\chi(M) = n + 1$, then M is either a hyperplane or a hyperquadric, the latter case arising only when n is odd.*

Proposition 2, combined with Lemma 1, implies

Theorem 4. *Let M be an n-dimensional complete intersection manifold imbedded in $P_{n+p}(C)$, $n > 1$. If $\chi(M) = n + 1$ and if $n + 1$ is a prime number, then M is a linear subspace.*

The following provides the answer to the case where codimension is 2.

Theorem 5. *Let M be an n-dimensional complete intersection manifold imbedded in $P_{n+2}(C)$. If $\chi(M) = n + 1$, then M is either a linear subspace or a hyperquadric in some hyperplane of $P_{n+2}(C)$, the latter case arising only when n is odd.*

Proof. Let $M = M_1 \cap M_2$ and $a_\alpha = \deg M_\alpha$ $(\alpha = 1, 2)$. Then it follows from Proposition 1 that

$$\chi(M) = \left[\sum_{k=0}^{n} (-1)^k \binom{n+3}{n-k} \sigma_k\right] a_1 a_2.$$

If $a_1 = 1$ or $a_2 = 1$, then Theorem 5 reduces to Proposition 2. Therefore it suffices to show that the Diophantine equation

$$\left[\sum_{k=0}^{n} (-1)^k \binom{n+3}{n-k} \sigma_k\right] a_1 a_2 = n + 1$$

has no solution satisfying $a_1 \geq 2$ and $a_2 \geq 2$, which is equivalent to the nonexistence of a solution to the Diophantine equation

$$\sum_{k=0}^{n} (-1)^k \binom{n+3}{n-k} \sigma_k = \xi \quad \text{for } 0 < \xi \leq (n+1)/4. \tag{1}$$

Multiplying both sides of (1) by $a_1 - a_2$, we obtain

$$\sum_{k=0}^{n} (-1)^k \binom{n+3}{n-k} (a_1^{k+1} - a_2^{k+1}) = \xi(a_1 - a_2),$$

which can be written as

$$\frac{(1-a_1)^{n+3} - 1 + (n+3)a_1}{a_1^2} + \xi a_1 = \frac{(1-a_2)^{n+3} - 1 + (n+3)a_2}{a_2^2} + \xi a_2. \tag{2}$$

Let

$$f_n(a) = ((1-a)^{n+3} - 1 + (n+3)a)/a^2 + \xi a.$$

Then it is not difficult to prove that

$$f_n'(a) > 0 \text{ (resp. } < 0) \quad \text{for } a \geq 2 \text{ if } n \text{ is odd (resp. even).} \tag{3}$$

This implies that $f_n(a)$ is monotonically increasing (resp. decreasing) for $a \geq 2$ if n is odd (resp. even). Therefore from (2) we deduce that $a_1 = a_2$.

Putting $a = a_1 = a_2$, from (1) we have

$$\sum_{k=0}^{n} (-1)^k (k+1) \binom{n+3}{n-k} a^k = \xi. \tag{4}$$

Let

$$g_n(a) = \sum_{k=0}^{n} (-1)^k (k+1) \binom{n+3}{n-k} a^k - \xi.$$

Then we have $f_n'(a) = -g_n(a)$, which, together with (3), implies that $g_n(a) < 0$ (resp. > 0) for $a \geq 2$ if n is odd (resp. even). Therefore (4) has no solution satisfying $a \geq 2$.

Thus we have proved that (1) has no solution satisfying $a_1 \geq 2$ and $a_2 \geq 2$. Q.E.D.

As an immediate consequence of Theorem 5 and Lemma 1, we have

Corollary. *Let M be an n-dimensional complete intersection manifold. If $\chi(M) = n + 1 = \nu_1\nu_2$ for some prime numbers ν_1 and ν_2, then M is either a linear subspace or a hyperquadric in some $(n+1)$-dimensional linear subspace, the latter case arising only when n is odd.*

4. **Low dimensional case.** In this section, we shall prove

Theorem 6. *Let M be an n-dimensional complete intersection manifold imbedded in $P_{n+p}(C)$. If $n \leq 52$ and $\chi(M) = n + 1$, then M is either a linear subspace or a hyperquadric in some $(n + 1)$-dimensional linear subspace, the latter case arising only when n is odd.*

Proof. From Theorem 4 and the Corollary, we see that Theorem 6 holds except when n is one of the following integers:

$$n = 7, 11, 15, 17, 19, 23, 26, 27, 29, 31, 35, 39, 41, 43, 44, 47, 49, 51.$$

Combining Theorems 2 and 5, we see that Theorem 6 is true for $n = 26$. For the cases where n is one of the following:

$$(*) \qquad n = 7, 11, 17, 19, 27, 29, 41, 43, 44, 49, 51,$$

we have $n + 1 = a_1 a_2 a_3$, where a_1, a_2 and a_3 are prime numbers. In all of these cases, if the smallest codimension is three, then we may assume that $\deg M_1 = a_1$, $\deg M_2 = a_2$ and $\deg M_3 = a_3$. By straightforward computations, we may easily find that

$$\sum_{k=0}^{n} (-1)^k \binom{n+4}{n-k} \sigma_k \neq 1.$$

Hence

$$\chi(M) = \left[\sum_{k=0}^{n} (-1)^k \binom{n+4}{n-k} \sigma_k \right] a_1 a_2 a_3 \neq n + 1.$$

This is a contradiction. Hence, for any integer n given in $(*)$, the smallest codimension p is ≤ 2. Therefore, by Theorem 5, we see that Theorem 6 holds for all such n. For the remaining cases, $n = 15, 23, 35, 39$ and 47, the smallest codimension p is ≤ 4 for $n = 15, 23, 35$ and 39 and is ≤ 5 for $n = 31$ and 47. In all cases, by a straightforward computation for $\Sigma_{k=0}^{n} (-1)^k \binom{n+p+1}{n-k} \sigma_k$, we see that the only possibilities are either a linear subspace or a hyperquadric. Q.E.D.

Remark. From the results above, we would like to state the following

Conjecture. The only n-dimensional complete intersection manifolds M in $P_{n+p}(C)$ with Euler-Poincaré characteristic $\chi(M) = n + 1$ are a linear subspace and a hyperquadric in some $(n + 1)$-dimensional linear subspace, the latter case arising only when n is odd.

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