DEFINING NORMAL SUBGROUPS
OF UNIPOTENT ALGEBRAIC GROUPS

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ABSTRACT. Let G be a connected unipotent algebraic group defined over the perfect field k. We show that polynomial generators \( x_1, \ldots, x_n \) for the ring \( k[G] \) can be chosen so that if \( N \) is any connected normal \( k \)-closed subgroup of \( G \), then \( I(N) \) can be generated by \( \text{codim } N \) \( p \)-polynomials in \( x_1, \ldots, x_n \) where \( p = \text{char } k \). Moreover \( k[G/N] \) can also be generated as a polynomial algebra over \( k \) by \( p \)-polynomials.

Introduction. These results are essentially an extension of a theorem of Rosenlicht [4, Theorem 1].

We use the notation and conventions of [1] throughout this paper.

Recall that a \( p \)-polynomial in \( k[T] \) is a linear form if \( p = 0 \) and a polynomial all of whose exponents are powers of \( p \) if \( p > 0 \). A \( p \)-polynomial in \( k[x_1, \ldots, x_n] \) is a sum of \( p \)-polynomials in each of the single variables \( x_1, \ldots, x_n \). A function \( f \in k[G] \) will be called additive if \( f(ab) = f(a) + f(b) \) for all closed points \( a, b \) in \( G \).

1. Frattini coordinates. Let \( G \) be a unipotent algebraic group. The Frattini subgroup of \( G \) is the intersection of all closed subgroups of codimension one. We shall denote this group by \( Fr(G) \).

Proposition 1. If \( G \) is a unipotent algebraic group then \( Fr(G) \) is a closed characteristic subgroup of \( G \). If \( G \) is connected and defined over the perfect field \( k \), then \( Fr(G) \) is connected and defined over \( k \). Moreover in the connected case \( G/\! Fr(G) \) has the structure of a vector group (over \( k \) if \( G \) is defined over \( k \)) and is the maximal such quotient.

Proof. The first assertion is immediate. Let \( H \subset G \) be a closed subgroup of \( G \) of codimension one. Since \( G/H \approx G_a \), \( H \) contains the commutator subgroup of \( G \) and the group generated by the \( p \)th powers of the elements of \( G \). It follows that \( Fr(G) \) also contains these subgroups.

Thus \( G/\! Fr(G) \) is connected, commutative and of exponent \( p \) hence by
Proposition 2] has the structure of a vector group. If \( N \subset G \) is any normal subgroup such that \( G/N \) is isomorphic to \( G'_{a}^{i} \) for some integer \( r \), then consider the natural map \( G \rightarrow G/N \cong G'_{a}^{i} \) followed by projection \( \Pi_{i}^{r} \) onto each factor, \( i = 1, 2, \cdots, r \). Each \( \Pi_{i}^{r} \) is a homomorphism with kernel say \( H_{i} \) and \( \bigcap H_{i} = N \). Since \( H_{i} \) has codimension one, \( N \supset Fr(G) \) and \( G/N \) is an image of \( G/Fr(G) \) which establishes the maximality assertion.

As for rationality and connectedness, let \( N \) be the closed normal subgroup generated by the commutator subgroup and \( p \)-th powers of the elements of \( G \). Then \( N \subset Fr(G) \), \( N \) is connected and \( G/N \) has the structure of a vector group [3, Proposition 2] so \( N = Fr(G) \). Since \( N \) is defined over \( k \) so is \( G/N \) [1, 6.8]. This completes the proof.

Now let \( G \) be a connected unipotent algebraic group and \( Fr(G) \) the Frattini subgroup of \( G \). If \( G \) is defined over the perfect field \( k \) then by [4, Corollary 2 of Theorem 1], \( k[G] \) is \( k \)-isomorphic to \( k[G/\text{Fr}(G)] \otimes k[\text{Fr}(G)] \). Let \( x_{1}, \cdots, x_{r} \) be additive coordinates for the vector \( k \)-group \( G/\text{Fr}(G) \) (cf. [3, §1]). Then \( k[G] = k[x_{1}, \cdots, x_{r}] \otimes k[\text{Fr}(G)] \). By the proposition \( F_1 = \text{Fr}(G) \) is again connected and defined over \( k \) and we may continue this process until we arrive at a complete set of polynomial generators for \( k[G] \). A set of polynomial generators \( x_{1}, \cdots, x_{n} \) obtained in this way will be called a set of Frattini coordinates for \( G \).

In case \( G \) itself has the structure of a vector group, these coordinates have essentially been studied by Rosenlicht [3], [4] and Tits [7, III, 3.3]. In particular the following proposition is easily deduced from their results.

**Proposition 2.** Let \( V \) be a connected unipotent algebraic group defined over the perfect field \( k \). Suppose \( V \) has the structure of a vector group over \( k \) and \( x_{1}, \cdots, x_{n} \) are Frattini coordinates for \( V \). Then

(i) if \( W \) is any \( k \)-closed subgroup of \( V \) then \( I(W) \) is generated by codim \( W \) \( p \)-polynomials in \( x_{1}, \cdots, x_{n} \);

(ii) the Frattini coordinates of \( k[V/W] \subset k[V] \) are \( p \)-polynomials in the Frattini coordinates of \( V \).

Now let \( G \) be any connected unipotent group defined over the perfect field \( k \). Let \( N \subset Fr(G) = F \) be a \( k \)-closed normal subgroup of \( G \). Then since \( G/N \cong G/F \times F/N \) we have \( k[G/N] \cong k[G/F] \otimes k[F/N] \). It follows from (ii) above that if \( Fr(F) = e \) then a set of Frattini coordinates for \( G/N \) may be taken to be \( p \)-polynomials in any fixed set of Frattini coordinates of \( G \).

**Theorem.** Let the connected unipotent algebraic group \( G \) be defined over the perfect field \( k \). Let \( x_{1}, \cdots, x_{n} \) be a fixed set of Frattini coordi-
nates for $G$. Suppose $Z$ is a closed connected central one dimensional subgroup of $G$ defined over $k$. Then there exists a set of Frattini coordinates of $G/Z$ in $R = k[G/Z] \subset k[G]$ which consists of $p$-polynomials in $x_1, \ldots, x_n$.

**Proof.** Let $F_0 = G \supset F_1 = F^r(G) \supset \cdots \supset F_s \supset e$ be the Frattini series of $G$. We argue by induction on the length, $s$, of the series. Thus, suppose $s = 1$. If $Z \subset F_1$ then $G/Z \cong G/F_1 \times F/Z$, hence $k[G/Z] \cong k[G/F_1] \otimes k[F/Z]$. By the remarks above, $k[G/Z]$ has $p$-polynomials in $x_1, \ldots, x_n$ as Frattini coordinates.

If $Z \cap F_1$ is finite we distinguish two cases.

Case 1. $Z \cap F_1 = e$. Then $ZF_1/F_1$ is a direct factor of $G/F_1$ and is not equal to $G/F_1$ since $Z$ is contained in a subgroup of codimension one.

Let $L \supset F_1$ be a connected $k$-closed subgroup of $G$ such that $L/F_1$ is a complement of $ZF_1/F_1$ in $G/F_1$ [3, Proposition 1]. Then $\text{codim } L = 1$, hence $L$ is normal in $G$. If $N = L \cap Z$ then $NF_1/F_1 = e$, hence $N \subset Z \cap F_1$. Thus $L \cap Z = e$ and clearly $LZ = G$.

Consider the commutative diagram

$$
\begin{array}{c}
ed \rightarrow F_1 \xrightarrow{i} G \xrightarrow{\pi} G/F_1 \rightarrow e \\
\downarrow m \quad \downarrow m' \\
Z \times L \xrightarrow{\nu} ZF_1/F_1 \times L/F_1 \rightarrow e
\end{array}
$$

where $i$ is inclusion, $\pi$ the quotient morphism, $m$ and $m'$ multiplication, and $\nu = \pi|_Z \times \pi|_L$.

We obtain a commutative diagram of Lie algebras:

$$
\mathfrak{L}(F_1) \xrightarrow{di} \mathfrak{L}(G) \xrightarrow{d\pi} \mathfrak{L}(G/F_1) \rightarrow 0
$$

$$
\begin{array}{c}
(\text{Ker } d(\pi/A)) \oplus \mathfrak{L}(F_1) \xrightarrow{d\alpha} \mathfrak{L}(Z) \oplus \mathfrak{L}(L) \xrightarrow{d\nu} \mathfrak{L}(ZF_1/F_1) \oplus \mathfrak{L}(L/F_1)
\end{array}
$$

Since $\alpha$ and $dm'$ are surjective so is $dm$. Thus $dm$ is an isomorphism and $m$ is separable. It follows from [1, Chapter II, 6.1] that $m : Z \times L \rightarrow G$ is an isomorphism.

Now choose new Frattini coordinates $y_1, \ldots, y_r, x_{r+1}, \ldots, x_n$ such that $V(y_1, \ldots, y_r) = F_1$ and $V(y_1) = L$. Then $y_1, \ldots, y_r$ are $p$-polynomials in $x_1, \ldots, x_r$ and $k[G/Z] = k[L] = k[y_1, \ldots, y_r, x_{r+1}, \ldots, x_n]$.

**Case 2.** $\Lambda = Z \cap F \neq e$. Then in $G/\Lambda$ we have the conditions of Case 1.
Hence \( k[G/Z] = k[G/A/Z/A] \subset k[G/A] \) is generated by \( p \)-polynomials in any set of Frattini coordinates of \( G/A \). But we may assume these last are \( p \)-polynomials in \( x_1, \cdots, x_n \). Hence \( k[G/Z] \) has a set of Frattini coordinates consisting of \( p \)-polynomials in \( x_1, \cdots, x_n \) and the case \( s = 1 \) is done.

If \( s > 1 \) we form the chain

\[
G \supset F_1' \supset F_1 \supset F_2' \supset F_2 \cdots \supset F_i' \supset Z \supset e
\]

where \( F_i'/Z \) is the \( i \)th term in the Frattini series of \( G/Z \), and \( F_i'/Z \) has the structure of a vector group over \( k \). Each \( F_i' \) may be taken to be connected, closed and defined over \( k \).

Suppose \( l \geq 2 \). Then \( F_1 \supset Z \). By induction \( k[F_1/Z] \subset k[F_1] \) has a set of Frattini coordinates which consists of \( p \)-polynomials in \( x_{r+1}, \cdots, x_n \). It then follows as before from the isomorphism \( k[G/Z] \cong k[G/F_1] \otimes k[F_1/Z] \) that \( G/Z \) has the desired property.

If \( l = 1 \) then \( F_1 \supset Z \). By induction \( k[F_1/Z] \subset k[F_1] \) has a set of \( p \)-polynomials in \( x_{r+1}, \cdots, x_n \). It then follows as before from the isomorphism \( k[G/Z] \cong k[G/F_1] \otimes k[F_1/Z] \) that \( G/Z \) has the desired property.

Corollary 1. Let \( G \) be a connected unipotent group defined over the perfect field \( k \). Let \( N \) be a connected closed normal subgroup of \( G \) also defined over \( k \). Then the Frattini coordinates of \( G/N \) in \( k[G/N] \) can be taken to be \( p \)-polynomials in any fixed set of Frattini coordinates for \( G \).

Proof. Any connected closed subgroup normal in \( G \) and defined over \( k \) contains a central connected subgroup of dimension one defined over \( k \) by [5]. The corollary now follows by induction on the dimension of \( N \).

Corollary 2. Suppose \( G \) and \( k \) are as above. Then every normal closed connected subgroup \( N \) of \( G \) which is defined over \( k \) can be defined by \( d = \text{codim } N \) \( p \)-polynomials in \( x_1, \cdots, x_n \). Moreover these may be chosen so as to generate the ideal \( I(N) \).

Proof. We have \( G/N \times N \cong G \) and by Corollary 1, \( k[G/N] \) is generated by \( p \)-polynomials in a fixed set of Frattini coordinates for \( G \). Say \( k[G/N] = k[l_1, \cdots, l_d] \subset k[G] \) where \( d = \text{codim } N \) and the \( l_i, i = 1, \cdots, d \), are \( p \)-polynomials in \( x_1, \cdots, x_n \). Then each \( l_i \) is constant on the fibres of \( \pi : G \rightarrow G/N \) and vanishes on \( N \).

Since \( k[G/N] \rightarrow k[G/N] \otimes k[N] = k[G] \) is a polynomial extension by [4,
Corollary 1 of Theorem 1] the ideal $(f_1, \cdots, f_d)k[G]$ is prime in $k[G]$. Hence $I(N) = (f_1, \cdots, f_d)k[G]$.

Remarks. 1. Corollary 2 is false without the assumption of normality on $N \triangleleft G$. Consider the following example suggested by Rosenlicht.

Let $G$ be the group of $3 \times 3$ upper triangular unipotent matrices

$$G = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in K, \ char \ K \neq 2 \right\}.$$

Let $x, y$ and $z$ be the obvious Frattini coordinates. Then

$$N = \left\{ \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} : t \in K \right\}$$

is a connected subgroup of $G$. The ideal $I(N) = (x - y, z - x^2/2)$ is clearly not generated by two $p$-polynomials.

Moreover if $H \triangleleft G$ is the subgroup defined by $x - y = 0$ and $N \triangleleft H$ is defined by $x^p - x = z - x^2/2 = 0$, then $N$ is a finite normal subgroup of $H$ which cannot be defined by two $p$-polynomials in the Frattini coordinates $x, z$ of $H$. Thus the assumption of connectivity is also necessary in Corollary 2.

2. If $G$ and $k$ are as in Theorem 1 and $H$ is any $k$-closed subgroup of codimension one (connected or not), then $H$ can be defined by a single $p$-polynomial in any set of Frattini coordinates. More generally, any $k$-closed subgroup of $G$ containing the Frattini subgroup of $G$ can be defined by $p$-polynomials. Simply note that $\text{codim}_G N = \text{codim}_{G/Fr(G)} N/Fr(G)$ and apply Proposition 2(i) and (ii).

3. An interesting application of Frattini coordinates is the following theorem of Sullivan.

**Theorem [6, Theorem 4].** A connected unipotent algebraic group defined over a field of characteristic $p > 0$ is conservative if and only if it has dimension one.

**Proof.** Recall that an algebraic group is conservative if the following condition holds.

Let $W$ be the group of all algebraic group automorphisms of $G$. If $f \in K[G]$ then $V_f = \{w_* (f) : w \in W\}$ is finite dimensional.

By [2, §1] this is equivalent to saying that $W$ may be given the structure of an algebraic group in such a way that the natural map $W \times G \to G$ is a morphism of varieties.

Now let $G$ be $1$ and $K[G] = K[x_1, \cdots, x_n]$, where $x_i, i = 1, \cdots, n,$
are Frattini coordinates for \( G \). Then it is easily checked (cf. [4, Corollary 2, p. 101]) that the assignments

\[
\begin{align*}
    x_i &\rightarrow x_i, & i = 1, \ldots, n - 1, \\
    x_n &\rightarrow x_n + P(x_1), & P \text{ a } p\text{-polynomial in } x_1,
\end{align*}
\]

give an automorphism of \( G \). In particular \( V_{x_n} \) is not finite dimensional. It is well known that \( \text{Aut}_{\text{Alg group}}(G_a) = G_m \) the multiplicative group.

4. If \( \text{char } K = 0 \), then with respect to the isomorphism of \( G \) with \( A^n \) given by a fixed set of Frattini coordinates, every normal subgroup is a linear subvariety.

5. The converse of Corollary 2 is easily seen to be false. If \( G \) is the group of Remark 1 above and \( H \) is the subgroup \( y = z = 0 \), it is easily seen that \( H \) is not normal in \( G \).

REFERENCES