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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706

Current address: Department of Mathematics, University of Alabama in Huntsville, Huntsville, Alabama 35807

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ON QUASINORMAL TOEPLITZ OPERATORS

ICHIRO AMEMIYA, TAKASHI ITO[1] AND TIN KIN WONG

ABSTRACT. It is proved in this note that a quasinormal Toeplitz operator is either normal or analytic.

1. **Introduction and the Theorem.** Let L^p be the Lebesgue space on the unit circle and let H^p be the corresponding Hardy space for $1 \le p \le +\infty$. The Toeplitz operator T_ϕ with symbol ϕ in L^∞ is the operator on H^2 defined by $T_\phi f = P(\phi f)$ for $f \in H^2$, where P is the orthogonal projection of L^2 onto H^2. A Toeplitz operator T_ϕ is analytic if its symbol ϕ is in H^∞. In this case T_ϕ is simply the multiplication on H^2 by ϕ with the multiplication on L^2 by ϕ as its normal extension. Thus any analytic Toeplitz operator is *subnormal*. We recall that an operator is subnormal if it is the restriction of some normal operator to one of its invariant subspaces. P. R. Halmos [2] raised the following interesting question: Is every subnormal Toeplitz operator either normal or analytic? The answer is not known yet. Some partial results are given in [3]. As a test question for this problem one may ask if quasinormal Toeplitz operators are either normal or analy-

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tic. We recall that an operator T is *quasinormal* if T commutes with T^*T, and also recall that every quasinormal operator is subnormal. The purpose of this note is to answer this test question.

Theorem. *A quasinormal Toeplitz operator is either normal or analytic.*

2. **Proof of the Theorem.** Before the proof we need to introduce several notations and need to show some identities which are fundamental for our proof.

The functions $e_n(z) = z^n$, $n = 0, \pm 1, \pm 2, \cdots$, form the usual orthonormal basis for L^2. The nth Fourier coefficient of f is denoted by $\hat{f}(n)$; $f = \sum_{n=-\infty}^{+\infty} \hat{f}(n)e_n$ in L^2. For elements g and h of H^2, $g \otimes h$ denotes the rank-one operator on H^2; $f \longrightarrow \langle f, g \rangle h$ where $\langle f, g \rangle$ is the inner product on H^2. The unilateral shift on H^2, namely, the multiplication by e_1, is denoted by U instead of T_{e_1}.

For $\phi \in L^\infty$, define

$$\phi_+ = U^* T_\phi e_0 = \sum_{n=0}^{\infty} \hat{\phi}(n+1)e_n \quad \text{and} \quad \phi_- = U^* T_\phi^* e_0 = \sum_{n=0}^{\infty} \overline{\hat{\phi}(-n-1)}e_n.$$

A simple computation (for example, by operating on the basis e_n) shows the following identities:

$$(1) \qquad T_\phi U - UT_\phi = \phi_- \otimes e_0, \qquad T_\phi^* U - UT_\phi^* = \phi_+ \otimes e_0.$$

Using (1) and its adjoint we have

$$(2)\ U^*T_\phi^* T_\phi U - T_\phi^* T_\phi = \phi_- \otimes \phi_-, \qquad U^* T_\phi T_\phi^* U - T_\phi T_\phi^* = \phi_+ \otimes \phi_+.$$

Let C_ϕ be the self-commutator of T_ϕ; $C_\phi = T_\phi^* T_\phi - T_\phi T_\phi^*$, it follows immediately from (2) that

$$(3) \qquad U^* C_\phi U - C_\phi = \phi_- \otimes \phi_- - \phi_+ \otimes \phi_+.$$

This identity shows that $\phi_- \otimes \phi_- = \phi_+ \otimes \phi_+$ is necessary and sufficient for T_ϕ to be normal. Because of (3), the necessity is trivial. Conversely, if $\phi_- \otimes \phi_- = \phi_+ \otimes \phi_+$, then $\phi_- = \gamma\phi_+$ for some number γ with $|\gamma| = 1$, that is $\overline{\hat{\phi}(-n)} = \gamma\hat{\phi}(n)$ for all $n \neq 0$. Therefore ϕ becomes a linear function of a real valued function. Hence T_ϕ clearly commutes with $T_\phi^* = T_{\overline{\phi}}$.

We add one more identity which we shall need later and which can be shown by combining (1) and (3).

$$(4)\ U^* C_\phi T_\phi U - C_\phi T_\phi = \phi_- \otimes U^* C_\phi e_0 + (\phi_- \otimes \phi_- - \phi_+ \otimes \phi_+)T_\phi.$$

We can now start our proof.

Proof of the Theorem. Let T_ϕ be quasinormal but let it be neither normal nor analytic. Under these assumptions, the first half of our argument will be showing that ϕ_- is an eigenvector of $T_\phi^* T_\phi$. Then using this fact, the rest of the argument will lead to a contradiction.

Since T_ϕ is quasinormal, that is $C_\phi T_\phi = 0$, the identity (4) becomes

$$\phi_- \otimes U^* C_\phi e_0 = (\phi_+ \otimes \phi_+ - \phi_- \otimes \phi_-)T_\phi. \tag{5}$$

The operator $S = \phi_+ \otimes \phi_+ - \phi_- \otimes \phi_-$ is a nonzero operator, because T_ϕ is not normal. Hence the range of S is the subspace spanned by ϕ_+ and ϕ_-. The adjoint of (5): $U^* C_\phi e_0 \otimes \phi_- = T_\phi^* S$ shows that T_ϕ^* maps ϕ_+ and ϕ_- into the one-dimensional subspace spanned by ϕ_-. Thus there are numbers α and β such that

$$T_\phi^* \phi_+ = \alpha\phi_- \quad \text{and} \quad T_\phi^* \phi_- = \beta\phi_-. \tag{6}$$

If we substitute (6) into the adjoint of (5), and note that $\phi_- \neq 0$ because T_ϕ is not analytic, then we come to

$$U^* C_\phi e_0 = \bar{\alpha}\phi_+ - \bar{\beta}\phi_-. \tag{7}$$

Using (1) and its adjoint, it follows that

$$T_\phi^* \phi_+ - T_\phi \phi_- = U^* C_\phi e_0 + \overline{\hat{\phi}(0)}\phi_+ - \hat{\phi}(0)\phi_-.$$

Substituting (6) and (7) into this equation, we have

$$T_\phi \phi_- = (\alpha + \bar{\beta} + \hat{\phi}(0))\phi_- - (\bar{\alpha} + \overline{\hat{\phi}(0)})\phi_+.$$

Operating this equation by T_ϕ^* and using (6) again, we conclude that

$$T_\phi^* T_\phi \phi_- = \lambda\phi_-, \tag{8}$$

where $\lambda = \alpha\beta - |\alpha|^2 + |\beta|^2 - \overline{\hat{\phi}(0)}\,\alpha + \hat{\phi}/(0)\beta$.

Now we can show that this will lead to a contradiction to the nonanalyticity of T_ϕ by arguing in the following way. Let K be the kernel of the operator $T_\phi^* T_\phi - \lambda I$. First, note that K is a nonzero and proper subspace of H^2. Because of $\phi_- \neq 0$, K is nonzero by (8) and K is proper by (2). K reduces T_ϕ because T_ϕ is quasinormal. Lastly, K is invariant under the backward shift U^*. To see this, we have

$$(T_\phi^* T_\phi - \lambda I)U^* = U^*(T_\phi^* T_\phi - \lambda I) - (T_\phi^* e_0) \otimes \phi_- - e_0 \otimes T_\phi^* \phi_+$$

by (1) and its adjoint. This equation and the fact that $T_\phi^* \phi_+ = \alpha\phi_-$ with $\phi_- \in K$ imply $(T_\phi^* T_\phi - \lambda I)^2 U^*(K) = \{0\}$. Hence $U^*(K) \subset K$.

The orthogonal complement $K^\perp$ of K now becomes a nonzero and proper subspace which is invariant under both the shift U and T_ϕ. The well-known Beurling's characterization of invariant subspaces of the shift U gives $K^\perp = \chi H^2$ for some nonconstant inner function χ. Since T_ϕ leaves χH^2 invariant, it is not hard to see that $T_\chi T_\phi = T_\phi T_\chi$. The commutativity of T_χ and T_ϕ implies that T_ϕ must be analytic [1], contradicting the assumption which we started with.

3. **Concluding remarks.** We conclude this note with two remarks.

Remark 1. It is easy to see that a quasinormal analytic Toeplitz operator is of the form $T_{\alpha\chi}$ where χ is an inner function and α is a number [3]. Here is a direct proof. If T_ϕ is quasinormal with $\phi \in H^\infty$, then the equation $T_\phi^* T_\phi T_\phi e_0 = T_\phi T_\phi^* T_\phi e_0$ implies that $\|T_\phi \phi\| = \|T_\phi^* \phi\|$ which in turn implies that $\| \, |\phi|^2 \| = \|P(|\phi|^2)\|$. So the real valued function $|\phi|^2$ is in H^2. Therefore $|\phi|^2 = \alpha^2$ for some constant α. Consequently $\phi = \alpha\chi$ with $\chi = \phi/|\phi|$ inner. The argument in the paragraph right after identity (3) shows that any normal Toeplitz operator must have a linear function of some real valued function as its symbol. This is of course the characterization of normal Toeplitz operators by Brown and Halmos [1]. The theorem can now be rephrased as follows:

The class of quasinormal Toeplitz operators is exactly the class of Toeplitz operators whose symbols are either linear functions of real valued functions or constant multiples of inner functions.

Remark 2. We consider the class of subnormal Toeplitz operators whose self-commutators are of rank one. In general, if T is an operator such that T^* acts as a scalar on the range of its self-commutator, that is

$$T^*(T^*T - TT^*) = \lambda(T^*T - TT^*)$$

for a number λ, then $T - \bar{\lambda} I$ is quasinormal. Note that $T - \bar{\lambda} I$ and T have the same self-commutator. In particular, if T is a subnormal operator with rank-one self-commutator then $T - \lambda I$ is quasinormal because T^* leaves invariant the range of its self-commutator for such subnormal T. It is then an immediate consequence of the Theorem that we have the following fact:

If T_ϕ is a subnormal Toeplitz operator with rank-one self-commutator, then its symbol ϕ is a linear function of some inner function χ, where $\chi(z) = (z - \alpha)/(1 - \bar{\alpha} z)$ for some $|\alpha| < 1$.

We note the inner function χ is of the form because the self-commutator C_ϕ is of rank one.

We are informed by the editor that K. Clancy has shown more generally that any pure subnormal operator with rank-one self-commutator is a linear function of the unilateral shift (see Indiana Univ. Math. J. 23 (1973), 497–511).

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DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY, TOKYO, JAPAN (Current address of Ichiro Amemiya)

DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MICHIGAN 48202 (Current address of Takashi Ito and Tin Kin Wong)