PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 50, July 1975

MORITA CONTEXTS OF ENRICHED CATEGORIES

J. FISHER-PALMQUIST AND P. H. PALMQUIST

ABSTRACT. Categories enriched over a closed category $\mathbf{V}$ are considered. The theorems and proofs are nonadditive while specializing when $\mathbf{V}$ is the category of abelian groups to yield different interpretations and proofs of old results. $\mathbf{V}$-adjoint equivalences of certain $\mathbf{V}$-functor categories are shown to correspond to generalized Morita equivalences between small $\mathbf{V}$-categories. Morita contexts are given a simple description as certain cospans and are shown to support a 2-dimensional structure.

For a bicomplete closed category $\mathbf{V}$ [6, p. 180] we show that our generalized $\mathbf{V}$-Morita equivalences between small $\mathbf{V}$-categories correspond to $\mathbf{V}$-adjoint equivalences between the corresponding $\mathbf{V}$-functor categories. $\mathbf{V}$-Morita equivalences are defined as Morita contexts invertible with respect to horizontal composition. $\mathbf{V}$-Morita contexts are a special kind of diagram $(\mathbf{C}_1 \to \dot{\mathbf{C}} \leftarrow \mathbf{C}_0)$ in the category of small $\mathbf{V}$-categories with horizontal composition induced by pushouts. In the classical case when $\mathbf{V}$ is the category of abelian groups and $\mathbf{C}_0$ and $\mathbf{C}_1$ are each additive categories with one object, our definition of Morita context is equivalent to that of Bass [1], [2]. Our point of view is to consider Morita contexts as arrows in a bicategory (à la Bénabou [3, pp. 3–6]) and to apply a morphism which takes a Morita context into a left adjoint. We use right Kan extensions [4] to express our basic constructions.

We use two special $\mathbf{V}$-categories $\mathbf{G}$ and $\mathbf{2}$, each of which has $\{0, 1\}$ as its set of objects, such that $\mathbf{G}(i, j) = I$ the unit object of $\mathbf{V}$ and $\mathbf{2}(i, j)$ is the terminal object of $\mathbf{V}$ for all i, j. (If $\mathbf{V}$ is cartesian closed, $\mathbf{G} = \mathbf{2}$.) The $\mathbf{V}$-category $[I]$ is the one object category with hom object I.

A $\mathbf{V}$-*Morita context* M is defined to be a pair $(\mathbf{C}, T\colon \mathbf{C} \to \mathbf{2})$ where $\mathbf{C}$

Presented to the Society, January 25, 1973 under the title *Morita contexts and the Brauer group of a closed category*; received by the editors July 20, 1973 and, in revised form, February 26, 1974.

AMS (MOS) subject classifications (1970). Primary 18D20, 18D25; Secondary 18A25, 18D05.

Key words and phrases. Closed category, enriched category, functor category, adjoint equivalence, Kan extension, cospan, dense, bicategory, 2-dimensional category, Morita context, Morita equivalence, atom.

is a small $\mathbf{V}$-category and T is a $\mathbf{V}$-functor, and a map of Morita contexts $F: (\mathbf{C}, T) \to (\mathbf{C}', T')$ is a $\mathbf{V}$-functor $F: \mathbf{C} \to \mathbf{C}'$ such that $T'F = T$. In other words, a Morita context $M = (\mathbf{C}, T)$ is a cospan $(\mathbf{C}_1 \xrightarrow{d_1} \mathbf{C} \xleftarrow{d_0} \mathbf{C}_0)$ in the category of small $\mathbf{V}$-categories [2] in which d_0 and d_1 are the inclusions of the corresponding *fibers* of T and the set of objects of $\mathbf{C}$ is the disjoint union of the sets of objects of $\mathbf{C}_0$ and $\mathbf{C}_1$, and a map of Morita contexts is a map of cospans. Morita contexts and their maps form a category $\mathbf{M}$ equipped with four important endofunctors. For $M = (\mathbf{C}, T)$, define the *transpose* of M by $M^t = (\mathbf{C}, \tau T)$ where $\tau: 2 \to 2$ is given by $\tau(i) = j$, $i \neq j$; define the *opposite* of M by $M^o = (\mathbf{C}^o, T^o)$. For a Morita context $M = (\mathbf{C}_1 \to \mathbf{C} \leftarrow \mathbf{C}_0)$ define the *left identity* of M by $l(M) = (\mathbf{C}_1 \otimes \mathbf{G}, L)$ where L is induced by projection on $\mathbf{G}$; define the *right identity* of M by $r(M) = l(M^t)$.

If $M = (\mathbf{C}_1 \to \mathbf{C} \leftarrow \mathbf{C}_0)$ and $M' = (\mathbf{C}_1' \to \mathbf{C}' \leftarrow \mathbf{C}_0')$ are Morita contexts with $r(M) = l(M')$, define the *$*$-composite* $M * M'$ by first taking the composite of the cospans, i.e., let

$$\begin{tikzcd} \mathbf{C}_0 = \mathbf{C}_1' \arrow[r] \arrow[d] & \mathbf{C}' \arrow[d] \\ \mathbf{C} \arrow[r] & \mathbf{C} \coprod_{\mathbf{C}_0} \mathbf{C}' \end{tikzcd}$$

be a pushout in the category of small $\mathbf{V}$-categories and obtain the cospan $(\mathbf{C}_1 \to \mathbf{C} \coprod_{\mathbf{C}_0} \mathbf{C}' \leftarrow \mathbf{C}_0')$, and then let the category of $M * M'$ be the full subcategory of $\mathbf{C} \coprod_{\mathbf{C}_0} \mathbf{C}'$ with objects the disjoint union of the objects of $\mathbf{C}_1$ and $\mathbf{C}_0'$. We note that for $X \in |\mathbf{C}_0'|$ and $Z \in |\mathbf{C}_1|$, $(\mathbf{C} \coprod_{\mathbf{C}_0} \mathbf{C}')(X, Z)$ is the coend [4] over all $Y \in |\mathbf{C}_0|$ of $\mathbf{C}(Y, Z) \otimes \mathbf{C}'(X, Y)$. If $\phi: M \to N$ and ϕ': $M' \to N'$ are maps of Morita contexts such that $r(\phi) = l(\phi')$ then there is a map $\phi * \phi': M * M' \to N * N'$ by the universal property of pushouts. This $*$-composition is associative up to isomorphism since the composition of cospans is. There are left and right identity isomorphisms $l_M: l(M) * M \to M$ and r_M: $M * r(M) \to M$.

Theorem 1. *If for small $\mathbf{V}$-categories $\mathbf{C}_0$ and $\mathbf{C}_1$ we define the category $\mathbf{B}(\mathbf{C}_0, \mathbf{C}_1)$ to have as objects Morita contexts $(\mathbf{C}_1 \to \mathbf{C} \leftarrow \mathbf{C}_0)$ and to have as maps only the maps of Morita contexts which are the identity on the fibers, then $*$-composition becomes a functor*

$$*: \mathbf{B}(\mathbf{C}_1, \mathbf{C}_2) \times \mathbf{B}(\mathbf{C}_0, \mathbf{C}_1) \to \mathbf{B}(\mathbf{C}_0, \mathbf{C}_2)$$

and $\mathbf{B}$ is a bicategory in the sense of Bénabou [3, pp. 3–6]. □

Define maps $\lambda_M: M * M^t \to l(M)$ and $\rho_M: M^t * M \to r(M)$ such that λ_M

is the identity on the fibers and λ_M on the other hom objects is induced by the compositions

$$\{\mathbf{C}(Y, Z) \otimes \mathbf{C}(X, Y) \to \mathbf{C}(X, Z) | Y \in \mathbf{C}_0\}$$

and $\rho_M = \lambda_{M^t}$. Ignoring associativity and left and right identity isomorphisms we have equations:

(1) $$(\lambda_M * M) = (M * \rho_M): M * M^t * M \to M;$$

(2) $$\lambda_{M*M'} = \lambda_M \cdot (M * \lambda_{M'} * M^t): (M * M') * (M * M')^t \to l(M);$$

(3) $$\lambda_N \cdot (\phi * \phi^t) = l(\phi) \cdot \lambda_M: M * M^t \to l(N);$$

for Morita contexts M, M', and N with $r(M) = l(M')$ and $\phi: M \to N$ a map of Morita contexts.

If $\mathbf{V}$ is the category of abelian groups, the correspondence between our Morita contexts and those of Newell [7] which are 4-tuples (U, V, μ, ν) is given as follows: $M = (\mathbf{C}_1 \to \mathbf{C} \leftarrow \mathbf{C}_0)$ corresponds to the 4-tuple with U (respectively, V) the restriction of the enriched hom of $\mathbf{C}$ to $\mathbf{C}_0^\circ \otimes \mathbf{C}_1$ (respectively, $\mathbf{C}_1^\circ \otimes \mathbf{C}_0$) and μ and ν the transformations induced by ρ_M and λ_M, respectively.

Theorem 2. *The following statements are equivalent for a Morita context* M.

(a) *There exists a Morita context* M' *such that* $M * M' \simeq l(M)$, *i.e.,* M *has a right* $*$*-inverse.*

(b) λ_M: $M * M^t \to l(M)$ *is a split epimorphism.*

(c) λ_M *is an isomorphism.*

(d) M^t *has a left* $*$*-inverse.*

(e) M° *has a right* $*$*-inverse.*

Proof. The only hard part is (a) $\Rightarrow$ (b) $\Rightarrow$ (c). To show (a) $\Rightarrow$ (b) we note that if ϕ: $M * M' \to l(M)$ is the isomorphism then equations (2) and (3) above give

$$\lambda_{l(M)} \cdot (\phi * \phi^t) = l(\phi) \circ \lambda_M \cdot (M * \lambda_{M'} * M^t).$$

Since $\lambda_{l(M)}$ is an isomorphism and so are $l(\phi)$ and $\phi * \phi^t$, we have λ_M is a split epimorphism.

If (b) holds then there is an s: $l(M) \to M * M^t$ such that $\lambda_M s = \mathrm{id}_{l(M)}$. Equation (1) applied to both M and M^t gives us that $s\lambda_M = (\lambda_M s) * M * M^t$ if we ignore all associativity and left and right identity isomorphisms. Hence $s\lambda_M = \mathrm{id}_{M*M^t}$. $\square$

A Morita context is said to be a *Morita equivalence* if it has both a left and a right $*$-inverse. Examples are $l(M)$ and $r(M)$.

There is another binary operation $\square$ on Morita contexts (which we might call vertical composition) which is always defined and is associative and commutative up to isomorphism. Namely, if $M = (\mathbf{C}_1 \to \mathbf{C} \leftarrow \mathbf{C}_0)$ and $N = (\mathbf{D}_1 \to \mathbf{D} \leftarrow \mathbf{D}_0)$ are Morita contexts $M \square N$ has as its category the full subcategory of $\mathbf{C} \otimes \mathbf{D}$ with fibers $\mathbf{C}_1 \otimes \mathbf{D}_1$ and $\mathbf{C}_0 \otimes \mathbf{D}_0$. For Morita contexts M, M', N and N' such that $r(M) = l(M')$ and $r(N) = l(N')$ we have equations:

$$(M \square N) * (M' \square N') = (M * M') \square (N * N'); \tag{4}$$

$$(M \square N)^t = M^t \square N^t \quad \text{and} \quad \rho_{M \square N} = \rho_M \square \rho_N. \tag{5}$$

Theorem 3. *Let M be a Morita context.*

(i) *If M is a Morita equivalence so are M^t and M°.*

(ii) *If M and M' are Morita equivalences with $r(M) = l(M')$, then $M * M'$ is a Morita equivalence.*

(iii) *If M and N have left $*$-inverses so does $M \square N$.*

(iv) *If $r(M) = r(N) = ([I] \to [I] \otimes \mathbf{G} \leftarrow [I])$ and $M \square N$ is a Morita equivalence, then M and N are Morita equivalences.*

Proof. Statements (i), (ii) and (iii) follow from Theorem 2 and equations (2) and (5). If the hypotheses of (iv) hold then equation (4) and the equalities $M = M \square r(N)$ and $N = r(M) \square N$ yield the equations $M \square N = (l(M) \square N) * M$ and $M \square N = (M \square l(N)) * N$, from which the conclusions of (iv) follow. $\square$

Let **Lad** be the 2-dimensional category with objects small $\mathbf{V}$-categories $\mathbf{C}$ and $\mathbf{Lad}(\mathbf{C}, \mathbf{C}')$ the category of $\mathbf{V}$-functors from $\mathbf{V}^{\mathbf{C}}$ to $\mathbf{V}^{\mathbf{C}'}$ which are $\mathbf{V}$-left adjoints, i.e., $\mathbf{V}$-cocontinuous, with maps $\mathbf{V}$-natural transformations. There is a strict homomorphism of bicategories Φ: $\mathbf{B} \to \mathbf{Lad}$ defined by

$$\Phi(\mathbf{C}) = \mathbf{C} \quad \text{and} \quad \Phi\left(\mathbf{C}_1 \xrightarrow{d_1} \mathbf{C} \xleftarrow{d_0} \mathbf{C}_0\right) = \mathrm{Ran}_{\mathbf{V}^{d_0}} \mathbf{V}^{d_1} = \mathbf{V}^{d_1} \cdot (\mathbf{V}^{d_0})^l,$$

where we have computed the right Kan extension in terms of $(\mathbf{V}^{d_0})^l$, the left adjoint of $\mathbf{V}^{d_0}$. Note that

$$\Phi(M)(\mathbf{C}_0(X, -))(Y) = \mathbf{C}(X, Y)$$

for X in $\mathbf{C}_0$ and Y in $\mathbf{C}_1$. We then have natural transformations

$$\Phi(\lambda_M)\colon \Phi(M) \cdot \Phi(M^t) \to \Phi(l(M)) = \mathrm{id}_{\mathbf{V}^{\mathbf{C}_1}}$$

and

$$\Phi(\rho_M)\colon \Phi(M^t) \cdot \Phi(M) \to \Phi(r(M)) = \mathrm{id}_{\mathbf{V}^{\mathbf{C}_0}}$$

and for X and Z in $\mathbf{C}_0$

(6) $$\Phi(\rho_M)(\mathbf{C}_0(X, -))(Z) = \rho_M(X, Z).$$

Theorem 4. *If*

$$M = \left(\mathbf{C}_1 \xrightarrow{d_1} \mathbf{C} \xleftarrow{d_0} \mathbf{C}_0\right)$$

*is a Morita context with a right *-inverse, then the following are true:*

(i) $\Phi(M^t)$ *is left adjoint to* $\Phi(M)$ *with counit* $\Phi(\rho_M)$ *and unit* $\Phi(\lambda_M)^{-1}$ *which is an isomorphism.*

(ii) *The functor* $\Phi(M^t)$ *maps* $\mathbf{V}$*-atoms* [5, (4.3)] *into* $\mathbf{V}$*-atoms and hence representables into* $\mathbf{V}$*-atoms.*

(iii) *The functor*

$$\mathbf{C}^{\circ} \xrightarrow{R} \mathbf{V}^{\mathbf{C}} \xrightarrow{\mathbf{V}^{d_0}} \mathbf{V}^{\mathbf{C}_0},$$

where R *is the Yoneda embedding, is* $\mathbf{V}$*-full and faithful, i.e.,* $d_0\colon \mathbf{C}_0 \to \mathbf{C}$ *is* $\mathbf{V}$*-codense* [4].

(iv) M *has a left *-inverse if and only if*

$$\mathbf{C}_1^{\circ} \xrightarrow{R_1} \mathbf{V}^{\mathbf{C}_1} \xrightarrow{\Phi(M^t)} \mathbf{V}^{\mathbf{C}_0}$$

is $\mathbf{V}$*-dense.*

Proof. (i) is a consequence of applying Φ which is a strict map of bicategories to equation (1) for M and M^t. (ii) follows from the fact that $\Phi(M)$ is the right adjoint of $\Phi(M^t)$ and is $\mathbf{V}$-cocontinuous. Thus for G a $\mathbf{V}$-atom in $\mathbf{V}^{\mathbf{C}_1}$, we have

$$\mathbf{V}^{\mathbf{C}_0}(\Phi(M^t)G, -) \simeq \mathbf{V}^{\mathbf{C}_1}(G, -) \cdot \Phi(M)$$

which is $\mathbf{V}$-cocontinuous. Part (iii) is equivalent to stating both that

$$\mathbf{C}_1^{\circ} \xrightarrow{R_1} \mathbf{V}^{\mathbf{C}_1} \xrightarrow{\Phi(M^t)} \mathbf{V}^{\mathbf{C}_0}$$

is $\mathbf{V}$-full and faithful, which is true since R_1 and $\Phi(M^t)$ are, and that

$$\Phi(M)(\mathbf{C}_0(C_0, -))C_1 = \mathbf{V}^{\mathbf{C}_0}(\Phi(M^t)R_1(C_1), \mathbf{C}_0(C_0, -))$$

for C_0 in $\mathbf{C}_0$ and C_1 in $\mathbf{C}_1$, which holds since $\Phi(M)$ is $\mathbf{V}$-right adjoint to $\Phi(M^t)$.

To show (iv) we note that since $\Phi(M^t)$ is $\mathbf{V}$-cocontinuous, $\Phi(M^t) \cdot R_1$ is $\mathbf{V}$-dense if and only if $\Phi(M^t)$ is $\mathbf{V}$-dense. But $\Phi(M^t)$ is $\mathbf{V}$-dense if and only if its right adjoint is full and faithful, i.e., $\Phi(\rho_M)$ is an isomorphism. But by (6) $\Phi(\rho_M)$ is an isomorphism if and only if ρ_M is one. □

Corollary 5. *The map of bicategories* Φ: **B** $\to$ **Lad** *induces an isomorphism of the Picard groupoids* [3, p. 57]

$$\hat{\Phi}: \mathrm{Pic}\,\mathbf{B} \to \mathrm{Pic}(\mathbf{Lad}). \quad \square$$

(Pic **B**)($\mathbf{C}_0$, $\mathbf{C}_1$) is the set of isomorphism classes of Morita equivalences, i.e., invertible arrows from $\mathbf{C}_0$ to $\mathbf{C}_1$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, IRVINE, CALIFORNIA 92664